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QUANTUM GROUPS OF DIMENSION pq^2

BY

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Dedicated to the memory of Professor S. A. Amitsur

ABSTRACT

In this paper we construct two families of non-trivial self-dual semisimple Hopf algebras of dimension pq^2 and investigate closely their (quasi) triangular structures. The paper contains also general results on finite-dimensional triangular Hopf algebras, unimodularity, semisimplicity and ribbon structures of finite-dimensional semisimple Hopf algebras.

Introduction

Let (A, R) be a triangular Hopf algebra. The category of A-modules in this case is very nice; it has a symmetry which makes it similar to the category of vector spaces. Some well known examples of such Hopf algebras are Sweedler's 4-dimensional Hopf algebra, H_4, which is neither commutative nor cocommutative, and the group algebra of a finite group G, which is cocommutative. Both H_4 and $k[G]$ are also pointed. That is, they are generated by a filtration induced by the group of grouplikes, $G(A)$. A natural question that arises is: Are all

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finite-dimensional triangular Hopf algebras pointed? This would imply that in characteristic 0 all such semisimple Hopf algebras are actually group algebras. It is this question that motivated Section 2. The answer is negative in general, but we conjecture it is true when A is so-called *minimal* triangular.

More generally, the connection between a Hopf algebra A and $G(A)$ has been at the root of many natural questions. Kaplansky has conjectured that if $\dim A = p$, p a prime, then $A = k[G(A)]$. This conjecture was recently proved by Zhu [Z]. Motivated by this, Masuoka [M1, M2] has completely characterized semisimple A of dimension $2p$, p^2 and p^3 via $G(A)$. We venture further and construct in this paper two new families of non-trivial semisimple Hopf algebras of dimension pq^2, where p and q are prime, via $G(A)$. These Hopf algebras, which contain a unique pq-dimensional sub-Hopf algebra, are not pointed and they are neither commutative nor cocommutative. For $q = 2$ they are even quasitriangular, and the members of one family are triangular for any p and q.

The purpose of this paper is two-fold: to study various properties of finite-dimensional triangular Hopf algebras, and to construct new quantum groups via biproducts.

The paper is organized as follows:

In Section 1 we recall some background material needed for this paper. We also prove some general properties of finite-dimensional quasitriangular Hopf algebras (see 1.3.1–1.3.6), and of the subcategory of biproducts (see 1.2.1–1.2.4) needed for this paper. For example:

THEOREM 1.3.5: *Let (A, R) be a finite-dimensional semisimple quasitriangular Hopf algebra over a field k of characteristic 0 or $p > (\dim A)^2$. Then (A, R) is ribbon.*

THEOREM 1.3.6: *Let (A, R) be a finite-dimensional cosemisimple quasitriangular Hopf algebra over any field k. Then A is unimodular.*

In Section 2 we discuss properties of finite-dimensional triangular Hopf algebras, (A, R), such as: connections between A and A^*, unimodularity and various connections between distinguished elements, and its effect on semisimplicity. For example:

THEOREM 2.2: *Let (A, R) be a finite-dimensional minimal triangular Hopf algebra over any field k. Then A and A^{*cop} are isomorphic as Hopf algebras,* *and $G(A)$ is abelian.*

THEOREM 2.8: *Let (A, R) be a finite-dimensional minimal triangular Hopf algebra over a field k of characteristic 0. If A is generated as an algebra by grouplikes and skew primitives, then $k[G(A)]$ admits a minimal triangular structure, and moreover $A = B \times k[G(A)]$ is a biproduct.*

THEOREM 2.13: *Let (A, R) be an odd-dimensional triangular Hopf algebra over a field k of characteristic 0. Then A and A^* are unimodular if and only if they are semisimple.*

The above properties will enable us to easily decide whether certain Hopf algebras admit triangular structures. These criteria will be used in Section 3.

In Section 3 we construct two families of non-trivial non-commutative non-cocommutative semisimple and cosemisimple Hopf algebras, A_{qp} and $\mathcal{A}_{qp}$, for any two prime numbers p and q satisfying $p = 1 \pmod q$. We prove that they are self-dual of dimension pq^2, give an explicit form of their sub-Hopf algebras and simple subcoalgebras, describe their groups of Hopf automorphisms and study questions of quasitriangularity. These families form counterexamples of various natural questions from Section 2, and some new non-trivial ribbon unimodular Hopf algebras which can be used to compute Hennings and Kauffman's links and 3-manifolds invariants. These families are biproducts of two quasitriangular Hopf algebras $B \times H$. As will be seen in Theorems 2.13 and 3.16, quasitriangularity of B and H will not imply quasitriangularity of $B \times H$ (obviously quasitriangularity of $B \times H$, for any B and H, implies quasitriangularity of H). Two of the main results of this section are:

THEOREM 2.13: *Let p and q be prime numbers satisfying $p = 1 \pmod q$, and let k be a field containing primitive* pth *and* q^2th *roots of unity. Then A_{qp} is quasitriangular if and only if $q = 2$. Furthermore, A_{2p} admits exactly $2p - 2$ minimal quasitriangular structures and exactly two non-minimal quasitriangular structures with $k[G(A_{2p})]$ as the corresponding minimal quasitriangular sub-Hopf algebra. Moreover, none of the above-mentioned quasitriangular structures is triangular.*

We also prove that A_{qp} and A_{qp}^{cop} are isomorphic as Hopf algebras if and only if $q = 2$.

THEOREM 3.16: *Let p and q be prime numbers satisfying $p = 1 \pmod q$, and let k be a field containing primitive pth and qth roots of unity. Then $\mathcal{A}_{qp}$ is a self-dual semisimple Hopf algebra of dimension pq^2 which is not isomorphic to A_{qp}. Moreover, $\mathcal{A}_{qp}$ admits a non-minimal triangular structure, with $k[G(\mathcal{A}_{qp})]$ as the corresponding minimal triangular sub-Hopf algebra, for any p and q. Furthermore, $\mathcal{A}_{qp}$ admits minimal quasitriangular structures if and only if $q = 2$, and $\mathcal{A}_{2p}$ admits exactly $2p - 2$ such structures none of which is triangular.*

In particular we show that $\mathcal{A}_{qp}$ and $\mathcal{A}_{qp}^{cop}$ are isomorphic as Hopf algebras for any p and q.

1. Preliminaries

We will focus on some background material, and prove some new general results needed in this paper. Throughout this paper k is a field and k^* is the group of units of k. The reader is referred to Sweedler's book [S] and Montgomery's book [M] as general references.

1.1 FINITE-DIMENSIONAL HOPF ALGEBRAS. Let A be a finite-dimensional Hopf algebra over k with antipode s. Then A is an A-bimodule under multiplication. Thus the transpose actions on A^*, described by

$$\langle a \rightharpoonup p, b\rangle = \langle p, ba\rangle \quad \text{and} \quad \langle p \leftharpoonup a, b\rangle = \langle p, ab\rangle$$

for $a, b \in A$ and $p \in A^*$, give A^* an A-bimodule structure. Similarly A is an A^*-bimodule, where

$$p \rightharpoonup a = \sum a_{(1)}\langle p, a_{(2)}\rangle \quad \text{and} \quad a \leftharpoonup p = \sum \langle p, a_{(1)}\rangle a_{(2)}$$

for $p \in A^*$ and $a \in A$, where we write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

"Twisting" multiplication and comultiplication in A gives rise to Hopf algebras A^{op} and A^{cop}, respectively. As a coalgebra $A^{op} = A$, and multiplication in A^{op} is defined by $a \cdot b = ba$ for $a, b \in A$. As an algebra $A^{cop} = A$, and comultiplication in A^{cop} is defined by $\Delta^{cop}(a) = \sum a_{(2)} \otimes a_{(1)}$ for $a \in A$. The antipode s is an algebra and a coalgebra anti-isomorphism. Thus $A^{op\,cop}$ is a Hopf algebra with antipode s, and A^{op}, A^{cop} are Hopf algebras with antipode s^{-1}. Thus $A \cong A^{op\,cop}$, and hence $A^{op} \cong A^{cop}$, as Hopf algebras.

Suppose that B is a Hopf algebra, and let $f\colon A \to B$ be a map of bialgebras. Then f is a map of Hopf algebras [S, Lemma 4.0.4].

A non-zero element g in A is said to be a **grouplike element** if $\Delta(g) = g \otimes g$. The set of grouplike elements of A is denoted by $G(A)$. $G(A)$ is finite and, by [NZ], the order of $G(A)$ divides $\dim A$. Since A is finite-dimensional $G(A^*) = \mathrm{Alg}_k(A, k)$. An element $x \in A$ is called a $g : h$ **skew primitive** if $\Delta(x) = x \otimes g + h \otimes x$, where $g, h \in G(A)$. If, moreover, $x \notin \mathrm{sp}_k\{g - h\}$ then x is called non-trivial. A is called **pointed** if its simple subcoalgebras are 1-dimensional, that is, they are generated by grouplike elements.

Let $\Lambda \in A$ be a non-zero left integral for A, and let $\lambda \in A^*$ be a non-zero right integral for A^*. The left integrals for A form a one-dimensional ideal of A. Hence there is a unique $\alpha \in G(A^*)$ such that $\Lambda a = \langle \alpha, a\rangle \Lambda$ for all $a \in A$. Likewise there is a unique $g \in G(A^{**}) = G(A)$ such that $p\lambda = \langle p, g\rangle \lambda$ for all $p \in A^*$. We call g **the distinguished grouplike element of** A and we call α **the distinguished grouplike element of** A^*. These grouplike elements play a fundamental role in the structure of A. A is said to be **unimodular** if the ideal of left integrals for A equals the ideal of right integrals for A. Thus A is unimodular if and only if $\alpha = \varepsilon$, and A^* is unimodular if and only if $g = 1$.

1.2 Biproducts. Let H be a Hopf algebra with antipode s_H over k and B a left H-module algebra with structure map $\tau\colon H \otimes B \to B$, usually written as $h \cdot b$. The well known **smash product** $B\#H$ is defined to be $B \otimes H$ as a vector space with multiplication

$$(1)\qquad (b\#h)(b'\#h') = \sum b(h_{(1)} \cdot b') \otimes h_{(2)}h'$$

for $b, b' \in B$ and $h, h' \in H$. Observe that $1_B\#1_H$ is the unity of $B\#H$ and that $j\colon B \to B\#H (b \mapsto b\#1)$ and $i\colon H \to B\#H (h \mapsto 1\#h)$ are algebra embeddings.

If moreover B is a left H-comodule coalgebra with structure map $\rho\colon B \to H \otimes B$ (we write $\rho(b) = \sum b^{(1)} \otimes b^{(2)}$) then one can define Δ on $B\#H$ by

$$(2)\qquad \Delta(b\#h) = \sum (b_{(1)}\#b_{(2)}^{(1)}h_{(1)}) \otimes (b_{(2)}^{(2)}\#h_{(2)})$$

and

$$(3)\qquad \varepsilon(b\#h) = \varepsilon(b)\varepsilon(h)$$

for $b \in B$ and $h \in H$. Observe that $\Pi\colon B\#H \to B (b\#h \mapsto b\varepsilon(h))$ and $\pi\colon B\#H \to H (b\#h \mapsto \varepsilon(b)h)$ are coalgebra surjections.

It was proved in [R1, Theorem 1] that $B\#H$ becomes a bialgebra with respect to the above if and only if

(i) $\Delta_B(1_B) = 1_B \otimes 1_B$,

(ii) ε_B is an algebra map,

(iii) $\Delta_B(bb') = \sum b_{(1)}(b_{(2)}^{(1)} \cdot b'_{(1)}) \otimes b_{(2)}^{(2)} b'_{(2)}$ for $b, b' \in B$,

(iv) $\sum (h_{(1)} \cdot b)^{(1)} h_{(2)} \otimes (h_{(1)} \cdot b)^{(2)} = \sum h_{(1)} b^{(1)} \otimes h_{(2)} \cdot b^{(2)}$ for $b \in B$ and $h \in H$,

(v) ρ is an algebra map and τ is a coalgebra map.

When $B\#H$ is a bialgebra as above we say that (H, B) is an **admissible pair**; we call it a **biproduct** and denote it by $B \times H$.

If, moreover, $I_B \in \mathrm{Hom}_k(B, B)$ has an inverse under convolution, s_B, then $B \times H$ is a Hopf algebra with antipode s given by

$$s(b \times h) = \sum (1 \times s_H(b^{(1)} h))(s_B(b^{(2)}) \times 1) \tag{4}$$

for $h \in H$ and $b \in B$ [R1, Proposition 2].

Note that $H \overset{i}{\hookrightarrow} B \times H \overset{\pi}{\to} H$ are Hopf algebra maps and $\pi \circ i = \mathrm{id}_H$. A is called a Hopf algebra with a projection if A contains a sub-Hopf algebra C such that $C \overset{i}{\hookrightarrow} A \overset{\pi}{\to} C$, where i is the inclusion map and π is a surjection of Hopf algebras satisfying $\pi \circ i = \mathrm{id}_C$. Thus $B \times H$ is such an A. In [R1, Theorem 3] it is shown that A is a Hopf algebra with a projection if and only if it is a biproduct. We use this description to prove Theorem 1.2.1 and Propositions 1.2.2 and 3.5.

In the following theorem we indicate that the subcategory of finite-dimensional biproducts is closed under taking duals.

THEOREM 1.2.1: *Let $A = B \times H$ be a finite-dimensional bialgebra over k with structure maps τ: $H \otimes B \to B$ and ρ: $B \to H \otimes B$. Then $A^* = B^* \times H^*$ with structure maps τ^*: $B^* \to H^* \otimes B^*$ and ρ^*: $H^* \otimes B^* \to B^*$.*

Proof: Follows directly from [R1, Theorems 2, 3]. ∎

In the following proposition we show that the subcategory of biproducts is closed under $\otimes$, "cop" and "op".

PROPOSITION 1.2.2: *Let $A = B \times H$ and $A' = B' \times H'$. Then: $A \otimes A'$, A^{cop} and A^{op} are biproducts.*

Proof: Let i: $H \hookrightarrow A$ and i': $H' \hookrightarrow A'$ be the inclusion maps, and π: $A \to H$ and π': $A' \to H'$ be the surjections. Recall that $\pi \circ i = \mathrm{id}_H$ and $\pi' \circ i' = \mathrm{id}_{H'}$. Now,

1. $i \otimes i'$: $H \otimes H' \hookrightarrow A \otimes A'$ is an inclusion, and $\pi \otimes \pi'$: $A \otimes A' \to H \otimes H'$ is surjective. Since $(\pi \otimes \pi') \circ (i \otimes i') = \mathrm{id}_{H \otimes H'}$, $A \otimes A'$ is a biproduct by [R1, Theorem 3].

2. Since i: $H^{cop} \hookrightarrow (B \times H)^{cop}$ is an inclusion, π: $(B \times H)^{cop} \to H^{cop}$ is surjective, and $\pi \circ i = \mathrm{id}_{H^{cop}}$, it follows that A^{cop} is a biproduct by [R1, Theorem 3].

3. Since i: $H^{op} \to (B \times H)^{op}$ is an inclusion, π: $(B \times H)^{op} \to H^{op}$ is surjective, and $\pi \circ i = \mathrm{id}_{H^{op}}$, it follows that A^{op} is a biproduct by [R1, Theorem 3].

This completes the proof of the proposition. ∎

In the following proposition we characterize some homomorphisms in the subcategory of biproducts.

PROPOSITION 1.2.3: *Let $B \times H$ and $B' \times H'$ be two biproducts over k with structure maps τ, ρ and τ', ρ' respectively. Suppose f: $B \to B'$ is an algebra and a coalgebra map, and g: $H \to H'$ is a bialgebra map. Then, the map $f \times g$: $B \times H \to B' \times H'$, given by $(f \times g)(b \times h) = f(b) \times g(h)$ for all $b \in B$ and $h \in H$, is a bialgebra map if and only if*

$$f(h \cdot b) = g(h) \cdot f(b) \quad \text{and} \quad \rho'(f(b)) = (g \otimes f)\rho(b)$$

for all $b \in B$ and $h \in H$.

Proof: Since on one hand

$$\begin{aligned}(f \times g)((b \times h)(b' \times h')) &= (f \times g)(\sum b(h_{(1)} \cdot b') \times h_{(2)}h') \\ &= \sum f(b)f(h_{(1)} \cdot b') \times g(h_{(2)})g(h')\end{aligned}$$

and on the other hand

$$\begin{aligned}((f \times g)(b \times h))((f \times g)(b' \times h')) &= (f(b) \times g(h))(f(b') \times g(h')) \\ &= \sum f(b)(g(h)_{(1)} \cdot f(b')) \times g(h)_{(2)}g(h') \\ &= \sum f(b)(g(h_{(1)}) \cdot f(b')) \times g(h_{(2)})g(h')\end{aligned}$$

for all $b, b' \in B$ and $h, h' \in H$, it follows that $f \times g$ is an algebra map if and only if

$$\sum f(h_{(1)} \cdot b) \times g(h_{(2)}) = \sum g(h_{(1)}) \cdot f(b) \times g(h_{(2)})$$

for all $b \in B$ and $h \in H$. The first condition is derived now by applying $I \otimes \varepsilon$ to both sides of the above equation. Similarly, one can show that the second condition is equivalent to $f \times g$ being a coalgebra map. This completes the proof of the proposition. ∎

Remark 1.2.4: Note that B' is a left H module via pull-back along g and that B is a left H' comodule via pull-back along g. Then, the conditions in Proposition 1.2.3 are equivalent to saying that f is also a module map and a comodule map.

1.3 QUASITRIANGULAR HOPF ALGEBRAS. We recall now the definition of a finite-dimensional quasitriangular Hopf algebra and some of its properties. We heavily use in the sequel the fact that there is a bijection between these Hopf algebras and certain associated Hopf algebra maps. We follow the conventions of [R3]. Let A be a finite-dimensional Hopf algebra over k and let $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$. Define a linear map $f_R : A^* \to A$ by $f_R(p) = \sum \langle p, R^{(1)} \rangle R^{(2)}$ for $p \in A^*$. The pair (A, R) is said to be a quasitriangular Hopf algebra if the following axioms hold ($r = R$):

(QT.1) $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,

(QT.2) $\sum \varepsilon(R^{(1)}) R^{(2)} = 1$,

(QT.3) $\sum R^{(1)} \otimes \Delta^{cop}(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes R^{(2)} \otimes r^{(2)}$,

(QT.4) $\sum R^{(1)} \varepsilon(R^{(2)}) = 1$ and

(QT.5) $(\Delta^{cop}(a)) R = R(\Delta(a))$ for all $a \in A$;

or equivalently, if $f_R : A^* \to A^{cop}$ is a Hopf algebra map and (QT.5) is satisfied.

Observe that (QT.5) is equivalent to

(QT.5)′ $\sum \langle p_{(1)}, a_{(2)} \rangle a_{(1)} f_R(p_{(2)}) = \sum \langle p_{(2)}, a_{(1)} \rangle f_R(p_{(1)}) a_{(2)}$
for all $p \in A^*$ and $a \in A$.

Note that the map f_R^*: $A^{*op} \to A$ is a Hopf algebra map which satisfies

$$f_R^*(p) = \sum \langle p, R^{(2)} \rangle R^{(1)} \qquad \text{and}$$
$$\sum \langle p_{(1)}, a_{(1)} \rangle a_{(2)} f_R^*(p_{(2)}) = \sum \langle p_{(2)}, a_{(2)} \rangle f_R^*(p_{(1)}) a_{(1)}$$

for all $p \in A^*$ and $a \in A$.

Conversely, let f: $A^{*cop} \to A$ be a Hopf algebra map, and let $R_f \in A \otimes A$ be the corresponding element via the canonical vector spaces isomorphism between $\mathrm{Hom}_k(A^*, A)$ and $A \otimes A$ (i.e. $f = f_{R_f}$). We say that f determines a quasitriangular structure on A if (A, R_f) is quasitriangular, or equivalently, if f satisfies (QT.5)′.

A quasitriangular Hopf algebra (A, R) is called **triangular** if $R^{-1} = R^{\tau}$ where $R^{\tau} = \sum R^{(2)} \otimes R^{(1)}$. Note that this is equivalent to $f_R * f_{R^{\tau}} = \varepsilon$ in the convolution algebra $\mathrm{Hom}_k(A^*, A)$, i.e. to $f_{R^{\tau}} = f_R \circ s$.

Let (A, R) be quasitriangular and suppose that $f : A \to A'$ is a surjective Hopf algebra map. Set $R' = (f \otimes f)(R)$; then (A', R') is quasitriangular.

Let (A, R) be quasitriangular. Set $R = \sum R^{(1)} \otimes R^{(2)}$, $B = \mathrm{sp}_k \{R^{(1)}\}$ and $H = \mathrm{sp}_k \{R^{(2)}\}$. Note that $B = \mathrm{Im}(f_R^*)$ and $H = \mathrm{Im}(f_R)$, hence B and H are sub-Hopf algebras of A. Let A_R be the sub-Hopf algebra of A generated by B and H. Then (A_R, R) is a quasitriangular Hopf algebra, there exists an isomorphism of Hopf algebras $f\colon B^{*cop} \to H$ and a unique surjection of Hopf algebras $F\colon D(B) \to A_R$ satisfying $F_{|B} = i_{|B}$ and $F_{|B^{*cop}} = f$, where $D(B)$ is the Drinfel'd double of B and i is the inclusion map [R3]. If $A = A_R$ then (A, R) is called a **minimal** quasitriangular Hopf algebra. We shall also say that A is a minimal (quasi)triangular Hopf algebra if there exists $R \in A \otimes A$ such that (A, R) is a minimal (quasi)triangular Hopf algebra.

Remark 1.3.1: Let (A, R) be a finite-dimensional quasitriangular Hopf algebra. If f_R is an isomorphism then (A, R) is minimal. Thus, if $f\colon A^{*cop} \to A$ is a Hopf algebra isomorphism satisfying (QT.5)$'$ then (A, R_f) is minimal quasitriangular. In particular, if A is commutative and cocommutative (e.g., $k[G]$ where G is a finite abelian group) and $f\colon A^* \to A$ is a Hopf algebra isomorphism, then (A, R_f) is minimal quasitriangular.

The converse of the above is not necessarily true, that is, (A, R) could be minimal quasitriangular without f_R being an isomorphism. In 2.2 we show that the converse holds when (A, R) is minimal triangular.

Let (A, R) be a finite-dimensional quasitriangular Hopf algebra with antipode s over k. For $\eta \in G(A^*)$ we define $g_\eta = \sum R^{(1)} \langle \eta, R^{(2)} \rangle = f_R^*(\eta)$. By [KR1], g_η is in the center of $G(A)$. Let $g \in A$ and $\alpha \in A^*$ be the distinguished grouplike elements of A and A^* respectively. As in [D] set

$$(5) \qquad u = \sum s(R^{(2)}) R^{(1)}, \quad h = g_\alpha g^{-1} \in G(A) \quad \text{and} \quad c = us(u).$$

Note that u, g_α and c belong to A_R. Since R is invertible it follows that u is invertible as well. By [D]

$$(6) \qquad \Delta(u) = (u \otimes u)(R^{\tau} R)^{-1}, \quad \varepsilon(u) = 1$$

and

$$s^2(a) = uau^{-1} \quad \text{for all } a \in A. \tag{7}$$

By (6), $u \in G(A)$ if and only if (A, R) is triangular.

The element c is a central element, hence called the Casimir element of (A, R). By [KR1]

$$c = u^2h, \tag{8}$$

thus by (5)

$$h = u^{-1}s(u) \in A_R \tag{9}$$

and so $g \in A_R$ as well. Since c is central, (7) and (8) imply that

$$s^4(a) = h^{-1}ah \quad \text{for all } a \in A. \tag{10}$$

The grouplike element $h = g_\alpha g^{-1}$ plays the primary role in the study of ribbon Hopf algebras [KR1].

A finite-dimensional ribbon Hopf algebra over k is a triple (A, R, v), where (A, R) is a finite-dimensional quasitriangular Hopf algebra over k and $v \in A$ satisfies the following:

(R.0) v is in the center of A,
(R.1) $v^2 = us(u)$,
(R.2) $s(v) = v$,
(R.3) $\varepsilon(v) = 1$,
(R.4) $\Delta(v) = (v \otimes v)(R^\tau R)^{-1} = (R^\tau R)^{-1}(v \otimes v)$.

Observe that $G = u^{-1}v$ is a grouplike element of A. It is called the **special grouplike element** of A. Ribbon Hopf algebras were introduced and studied by Reshetikhin and Turaev [RT]. The reader is referred to [R2, Section 2] and [H, K, KR1, KR2] for an extensive study of ribbon Hopf algebras and their connections with Hennings and Kauffman's invariants.

Remark 1.3.2: Any triangular Hopf algebra is ribbon with 1 as the ribbon element and u^{-1} as the special grouplike element.

The following lemma connects u to unimodularity and thus has interesting corollaries.

LEMMA 1.3.3: *Let (A, R) be a finite-dimensional quasitriangular Hopf algebra over k and let $\tilde{g}$ and $\tilde{\alpha}$ be the distinguished grouplike elements of A_R and A_R^* respectively. Then:*

(1) $g_\alpha g^{-1} = g_{\tilde{\alpha}}\tilde{g}^{-1}$, *hence* $g_{\tilde{\alpha}}^{-1}g_\alpha$ *does not depend on* R.

(2) *Suppose further that* A_R *and* A_R^* *are unimodular. Then* $u = s(u)$, $s^4 = I$ *and* $g_\alpha = g$.

Proof: (1) Set $\tilde{h} = g_{\tilde{\alpha}}\tilde{g}^{-1}$. By the above constructions, and since $u \in A_R$, we have $\tilde{h} = u^{-1}s(u)$, and thus $\tilde{h} = h$.

(2) If A_R and A_R^* are unimodular then $\tilde{g} = 1$ and $\tilde{\alpha} = \varepsilon$, hence $g_{\tilde{\alpha}} = 1$, and so $\tilde{h} = 1$. By (1), $h = 1$ and hence $s^4 = I$, $u = s(u)$ and $g_\alpha = g$. ∎

Remark 1.3.4: It may happen that A_R will be unimodular, but A_R^* will not. Let A be Sweedler's example. Recall that $D(A)$ is minimal quasitriangular, and unimodular [R3, Theorem 4]. But, since A and A^* are not unimodular, $D(A)^*$ is not unimodular [R3, Corollary 4].

A corollary of 1.3.3 is that under semisimplicity assumptions u has further properties.

THEOREM 1.3.5: *Let* (A, R) *be a finite-dimensional quasitriangular Hopf algebra over* k. *Then:*

(1) *If* A_R *is semisimple, then* $u = s(u)$ *and* $s^4 = I$.

(2) *Assume the characteristic of* k *is 0 or* $p > (\dim A)^2$. *If* A *is semisimple then* A *is ribbon with* u *as the ribbon element and 1 as the special grouplike element.*

Proof: (1) Since A_R is minimal quasitriangular it follows that it is also cosemisimple [R3, Proposition 14], and hence that A_R and A_R^* are unimodular. Thus part (1) follows from Lemma 1.3.3.

(2) Since a sub-Hopf algebra of a semisimple Hopf algebra is also semisimple it follows that A_R is semisimple, and hence that $u = s(u)$ by part (1). Since A_R is also cosemisimple, it follows from our assumption on the ground field, and [LR2, Theorem 3], that $s^2 = I$, and hence that u is a central element of A. This concludes the proof of the corollary. ∎

Let A be a finite-dimensional Hopf algebra over k. Recall [LR1, Theorem 3.3] that over characteristic 0, if A is cosemisimple then it is also semisimple and hence unimodular. In the following corollary of Lemma 1.3.3 we prove that for quasitriangular A, unimodularity follows regardless of characteristic.

THEOREM 1.3.6: *Let* (A, R) *be a finite-dimensional cosemisimple quasitriangular Hopf algebra over any field* k. *Then* A *is unimodular.*

Proof: By [R3, Proposition 14] A_R is a semisimple and cosemisimple quasitriangular Hopf algebra. It follows from Lemma 1.3.3 that $g_\alpha = g$. But A^* is semisimple, hence $g = 1$ and so $g_\alpha = 1$.

Now recall that the map $F\colon D(A) \to A$ given by $F(p \bowtie a) = f_R(p)a$ for all $p \in A^*$ and $a \in A$ is a projection of Hopf algebras [D]. Let λ and Λ be non-zero left integral and right integral for A^* and A respectively; then by [R3, Theorem 4] $\lambda \bowtie \Lambda$ is a two-sided integral of $D(A)$. Now, since A^* is semisimple, $\langle \lambda, 1\rangle \neq 0$ and so

$$\begin{aligned} F(\lambda \bowtie \Lambda) &= \sum \langle \lambda, R^{(1)}\rangle R^{(2)}\Lambda \\ &= \sum \langle \lambda, R^{(1)}\rangle \langle \alpha, R^{(2)}\rangle \Lambda \\ &= \langle \lambda, g_\alpha\rangle \Lambda = \langle \lambda, 1\rangle \Lambda \end{aligned}$$

is a non-zero two-sided integral of A, and we are done. ∎

2. Triangular Hopf algebras

In this section we discuss properties of finite-dimensional triangular Hopf algebras such as: connections between A and A^*, unimodularity and various connections between the distinguished elements, and its effect on semisimplicity. These properties will enable us to easily decide whether certain Hopf algebras admit triangular structures. These criteria will be used in Section 3.

Radford has proved that if A is a finite-dimensional Hopf algebra then $D(A)^*$ is quasitriangular if and only if both A and A^* are. Moreover, if (A, r) and (A^*, R) are quasitriangular then $(D(A)^*, \mathcal{R})$ is quasitriangular where

$$\mathcal{R} = \sum (h_i \cdot r^{(2)} \otimes R^{(1)} h_j^*) \otimes (r^{(1)} \cdot s^{-1}(h_j) \otimes h_i^* R^{(2)}) \tag{11}$$

and $\{h_i\}$ and $\{h_i^*\}$ are dual bases of H and H^* respectively [R3, page 311]. In the following we check what happens in the triangular case.

PROPOSITION 2.1: *Suppose A is a finite-dimensional Hopf algebra over k. Then the following are equivalent:*

(1) *$D(A)^*$ admits a triangular structure.*
(2) *A and A^* admit triangular structures.*

Proof: Since A^{op} and A^* are homomorphic images of $D(A)^*$, (1) implies (2). Suppose now that (A, r) and (A^*, R) are triangular and let f_r and f_R be the corresponding maps. Then it is not hard to verify that the map $F_{\mathcal{R}}: D(A)^{cop} \to D(A)^*$ corresponding to $\mathcal{R}$ in (11) is given by

$$F_{\mathcal{R}}(p \bowtie h) = \sum s^{-1}(h_{(2)}) f_r^*(p_{(1)}) \otimes p_{(2)} f_R(h_{(1)}) \tag{12}$$

and that the map $F_{\mathcal{R}}^* = F_{\mathcal{R}^\tau}: D(A)^{op} \to D(A)^*$ is given by

$$F_{\mathcal{R}}^*(p \bowtie h) = \sum f_r(p_{(2)}) h_{(1)} \otimes f_R^*(h_{(2)}) s^{-1}(p_{(1)}). \tag{13}$$

We prove that part (2) implies part (1) by showing that $F_{\mathcal{R}\mathcal{R}^\tau} = F_{\mathcal{R}} * F_{\mathcal{R}^\tau} = \varepsilon_{D(A)}$. Indeed,

$$\begin{aligned}
&(F_{\mathcal{R}} * F_{\mathcal{R}^\tau})(p \bowtie h) \\
&= (F_{\mathcal{R}} \otimes F_{\mathcal{R}^\tau}) \sum (p_{(2)} \bowtie h_{(1)} \otimes p_{(1)} \bowtie h_{(2)}) \\
&= \sum (s^{-1}(h_{(2)}) f_r^*(p_{(3)}) \otimes p_{(4)} f_R(h_{(1)})) \cdot (f_r(p_{(2)}) h_{(3)} \otimes f_R^*(h_{(4)}) s^{-1}(p_{(1)})) \\
&= \sum f_r(p_{(2)}) h_{(3)} s^{-1}(h_{(2)}) f_r^*(p_{(3)}) \otimes p_{(4)} f_R(h_{(1)}) f_R^*(h_{(4)}) s^{-1}(p_{(1)}) \\
&= \sum f_r(p_{(2)}) f_r^*(p_{(3)}) \otimes p_{(4)} f_R(h_{(1)}) f_R^*(h_{(2)}) s^{-1}(p_{(1)}) \\
&= \sum \varepsilon(p_{(2)}) 1 \otimes p_{(3)} \varepsilon(h) s^{-1}(p_{(1)}) \\
&= \varepsilon(p \bowtie h).
\end{aligned}$$

This concludes the proof of the theorem. ∎

Let (A, R) be a finite-dimensional quasitriangular Hopf algebra. It is well known that $G(A^*)$ is abelian [D]. In the following we show that when (A, R) is minimal triangular, much more can be said; the converse of 1.3.1 is true.

THEOREM 2.2: *Let (A, R) be a finite-dimensional minimal triangular Hopf algebra over k. Then:*

1. *The map $f_R: A^{*cop} \to A$ is an isomorphism of Hopf algebras. In particular A^* and $D(A)^*$ admit triangular structures.*
2. *The groups $G(A)$ and $G(A^*)$ are abelian.*

Proof: (1) Let $B = \mathrm{sp}_k\{R^{(1)}\}$ and $H = \mathrm{sp}_k\{R^{(2)}\}$. Then $B \cong H^{*cop}$ as Hopf algebras. By [D], $R^{-1} = \sum s(R^{(1)}) \otimes R^{(2)}$, and hence in the triangular case $\sum s(R^{(1)}) \otimes R^{(2)} = \sum R^{(2)} \otimes R^{(1)}$. Thus, $B = H$. Since $A = A_R = BH$ we conclude that $A = H^2 = H$ and $A \cong A^{*cop}$ via f_R, hence A^* admits a triangular structure.

(2) In general, if (A,R) is quasitriangular then $G(A^*)$ is abelian [D]. Since f_R is an isomorphism of Hopf algebras it follows that $G(A) \cong G(A^*)$, and hence $G(A)$ is abelian. ∎

The following are some consequences of Theorem 2.2 which shed light on minimal triangular pointed or semisimple Hopf algebras.

THEOREM 2.3: *Suppose (A,R) is a finite n-dimensional semisimple minimal triangular Hopf algebra over a field k containing a primitive nth root of unity and that $G(A)$ is cyclic. Then $G(A) = \langle u \rangle$, and it is trivial or of order 2.*

Proof: Since (A,R) is semisimple and minimal it is also cosemisimple, and hence A and A^* are unimodular. Thus, by Lemma 1.3.3, $u = s(u)$. But $u \in G(A)$, hence $s(u) = u^{-1}$, and we have $u^2 = 1$. Since (A,R) is triangular it follows that $f_R^* = f_R \circ s$ and hence

$$\begin{aligned}\langle \alpha, u\rangle &= \sum \langle \alpha, s(R^{(2)})R^{(1)}\rangle = \sum \langle \alpha, s(R^{(2)})\rangle\langle \alpha, R^{(1)}\rangle \\ &= \langle \alpha, s(f_R(\alpha))\rangle = \langle \alpha, f_R(s^{-1}(\alpha))\rangle = \langle \alpha, f_R(s(\alpha))\rangle \\ &= \langle \alpha, f_R^*(\alpha)\rangle = \langle \alpha, f_R(\alpha)\rangle\end{aligned}$$

for $\alpha \in G(A^*)$. By Theorem 2.2, f_R induces an isomorphism between $G(A)$ and $G(A^*)$, hence $G(A^*)$ is cyclic too. Let β be a generator of $G(A^*)$; then $f_R(\beta)$ generates $G(A)$ and we have by the above that $\langle \beta, u\rangle = \langle \beta, f_R(\beta)\rangle$. Now, generally if $G = (x)$ is cyclic of order m then $\hat{G}$ (the group of characters) is cyclic of order m generated by η, and $\langle \eta, x\rangle$ is a primitive mth root of unity. Thus, if $u = 1$ then, by the above, $\langle \beta, f_R(\beta)\rangle = \langle \beta, u\rangle = 1$ so $m = 1$. If $u \neq 1$ then, since $u^2 = 1$, $1 = \langle \beta, u^2\rangle = \langle \beta, u\rangle^2 = \langle \beta, f_R(\beta)\rangle^2$ hence $\langle \beta, f_R(\beta)\rangle = -1$, and we have $m = 2$. This completes the proof of the theorem. ∎

Remark 2.4: We shall use the above theorem in Corollary 3.10 to prove that certain Hopf algebras we construct, $A = A_{2p}$, are never triangular, though A is minimal quasitriangular, $A \cong A^{*cop}$ as Hopf algebras and $G(A)$ is abelian.

The following is an example of a semisimple minimal triangular Hopf algebra, A, such that $G(A)$ is not cyclic.

Example 2.5: Suppose that the field k contains a primitive nth root of unity, ω, and let $A = k[\langle a\rangle] \otimes k[\langle b\rangle] \cong k[Z_n \times Z_n]$. Then $G(A)$ is not cyclic but A admits a minimal triangular structure. To see this let $\alpha, \beta \in A^*$ be so that $\langle \alpha, a^i b^j\rangle = \omega^j$

and $\langle\beta, a^ib^j\rangle = \omega^i$. Then $A^* = k[\langle\alpha\rangle] \times k[\langle\beta\rangle]$, and the map $f\colon A^* \to A$, given by $f(\alpha) = a$ and $f(\beta) = b^{-1}$, determines a minimal quasitriangular structure on A. Moreover, $\langle\alpha^i\beta^j, u\rangle = \sum\langle\alpha^i\beta^j, s(R^{(2)})R^{(1)}\rangle = \sum\langle\alpha^i\beta^j, s(R^{(2)})\rangle\langle\alpha^i\beta^j, R^{(1)}\rangle = \langle\alpha^i\beta^j, (s\circ f)(\alpha^i\beta^j)\rangle = \langle\alpha^i\beta^j, s(a^ib^{-j})\rangle = \langle\alpha^i\beta^j, a^{-i}b^j\rangle = \omega^{ij-ij} = 1$, hence $u = 1$, which implies that A is triangular.

Since the standard finite-dimensional triangular Hopf algebras are pointed, a natural question is whether this is always true. The following example shows the answer is negative.

Example 2.6: Let H be Sweedler's 4-dimensional Hopf algebra, and suppose that the characteristic of k is not 2. Then $D(H)^*$ is triangular but never minimal. To see this recall [G2, R3], which describes all the quasitriangular structures that H admits. Note that all of them are triangular. Since H^* and H are isomorphic as Hopf algebras, it follows from Theorem 2.1 that $D(H)^*$ admits triangular structures. Recall that H and H^* are pointed and hence that $D(H)^{cop}$ is pointed too. Since $D(H)^*$ is not pointed [R1, page 315] it is not isomorphic to $D(H)^{cop}$ as a Hopf algebra. Therefore we conclude from Theorem 2.2 that $D(H)^*$ is never minimal.

The examples in Section 3 (Theorem 3.16) show that even under semisimplicity the answer is still negative. However, A in these examples is not *minimal* triangular. We conjecture:

CONJECTURE 2.7: *Let (A, R) be a finite-dimensional minimal triangular Hopf algebra over a field k of characteristic 0. Then A is pointed. In particular, if A is semisimple then $A = k[G(A)]$ is commutative.*

One instance in which A is assured to be pointed is when A is generated by grouplike elements and skew primitives (all standard examples, e.g. $U_q(sl_n)'$, $U_{(N,\nu,\omega)}$ and H_4, are of this type [G1, GW]). We then prove:

THEOREM 2.8: *Let (A, R) be a finite-dimensional minimal triangular Hopf algebra over a field k of characteristic 0. If A is generated as an algebra by grouplike elements and skew primitive elements then:*

(1) *$k[G(A)]$ admits a minimal triangular structure.*
(2) *There exists a projection $\pi\colon A \to k[G(A)]$, thus $A = B \times k[G(A)]$ is a biproduct.*
(3) *If $A \neq k1$ and $G(A)$ is cyclic, then $G(A) = \langle u\rangle$ is of order 2.*

Proof: (1) By Theorem 2.2, $G(A)$ is abelian hence $k[G(A)]^* \cong k[G(A)]$. By the same theorem $k[G(A)] \cong k[G(A^{*cop})]$, hence $\dim k[G(A)]^* = \dim k[G(A^*)]$. Since A and A^{*cop} are isomorphic as Hopf algebras, it follows that A^* is also minimal triangular. Suppose (A^*, r) is minimal triangular and consider the following series of maps:

$$k[G(A)] \overset{i}{\hookrightarrow} A^{cop} \xrightarrow{f_r} A^* \xrightarrow{i^*} k[G(A)]^*.$$

We claim that $i^*_{|k[G(A^*)]}: k[G(A^*)] \to k[G(A)]^*$ is injective, hence an isomorphism of Hopf algebras. Indeed, if $\alpha, \beta \in G(A^*)$ are such that $i^*(\alpha) = i^*(\beta)$ then $\langle \alpha, g\rangle = \langle \beta, g\rangle$ for all $g \in G(A)$, but $\langle \alpha, x\rangle = \langle \beta, x\rangle = 0$ for all non-trivial skew primitives x (if such x does not exist then, by our assumption, $A = k[G(A)]$ and there is nothing to prove). Hence $\alpha = \beta$ on generators of A, thus $\alpha = \beta$ on A. We thus conclude that the map $i^* \circ f_r \circ i: k[G(A)] \to k[G(A)]^*$ is an isomorphism of Hopf algebras, hence determines a minimal quasitriangular structure on $k[G(A)]^*$ by Remark 1.3.1. We now wish to show that this structure is minimal *triangular*. Indeed, $(k[G(A)]^*, (i^* \otimes i^*)(r))$ is triangular and

$$\begin{aligned} f_{(i^*\otimes i^*)(r)}(g) &= \sum \langle g, i^*(r^{(1)})\rangle i^*(r^{(2)}) \\ &= i^*(\sum \langle i(g), r^{(1)}\rangle r^{(2)}) \\ &= i^* \circ f_r \circ i(g) \end{aligned}$$

for all $g \in k[G(A)]$. This implies that

$$f_{(i^*\otimes i^*)(r)} = i^* \circ f_r \circ i,$$

hence $(k[G(A)]^*, (i^* \otimes i^*)(r))$ is minimal triangular. Since $k[G(A)]$ and $k[G(A)]^*$ are isomorphic as Hopf algebras we are done.

(2) Set $\varphi = i^* \circ f_r \circ i$ and $\pi = \varphi^{-1} \circ i^* \circ f_r$. Then $\pi: A \to k[G(A)]$ is onto, and moreover $\pi \circ i = \varphi^{-1} \circ i^* \circ f_r \circ i = \varphi^{-1} \circ \varphi = \mathrm{id}_{k[G(A)]}$, hence π is a projection of Hopf algebras and we are done.

(3) Suppose $A \neq k1$ and $G(A)$ is cyclic. By our assumption on the ground field, $k[G(A)]$ is semisimple and, by part (1), it is also minimal triangular. Therefore, by Theorem 2.3, $G(A) = \langle u\rangle$ is trivial or of order 2. Since $A \neq k1$, if $G(A)$ were trivial A would contain a primitive element, which is impossible in characteristic 0. Thus, $G(A) = \langle u\rangle$ is of order 2 and we are done. ∎

If A is not semisimple then it is not necessarily unimodular. This lack of unimodularity gives rise to the distinguished grouplikes, which in turn affect

s^2. In the following we describe connections between these elements of A and A_R, which turn out to be related to u. The results are tight when A is odd-dimensional.

LEMMA 2.9: *Suppose A and B are finite-dimensional Hopf algebras over k and let $f\colon A \to B$ be an isomorphism of Hopf algebras. Then $f^*(\beta) = \alpha$ where α and β are the distinguished grouplike elements of A^* and B^* respectively.*

Proof: Let λ be a non-zero left integral of A. Then $f(\lambda)$ is a non-zero left integral of B. Hence, $f(\lambda)b = \langle\beta, b\rangle f(\lambda)$ for all $b \in B$. Write $b = f(a)$ for $a \in A$. Then $f(\lambda)f(a) = \langle\beta, f(a)\rangle f(\lambda) = \langle f^*(\beta), a\rangle f(\lambda)$ for all $a \in A$. On the other hand, $f(\lambda)f(a) = f(\lambda a) = f(\langle\alpha, a\rangle\lambda) = \langle\alpha, a\rangle f(\lambda)$ for all $a \in A$. Thus the result follows. ∎

COROLLARY 2.10: *Suppose (A, R) is a finite-dimensional quasitriangular Hopf algebra over k and that $f_R\colon A^{*cop} \to A$ is an isomorphism of Hopf algebras. Let g and α be the distinguished grouplike elements of A and A^* respectively. Then:*

(1) $g_\alpha = g^{-1}$, *hence* $h = g^{-2}$.

(2) $s(u) = ug^{-2}$.

Proof: Since $g_\alpha = f_R^*(\alpha)$ and g^{-1} is the distinguished grouplike element of $(A^{*cop})^* = A^{op}$, part (1) follows from Lemma 2.9. Part (2) follows now from part (1) and (9). ∎

THEOREM 2.11: *Suppose (A, R) is a finite-dimensional triangular Hopf algebra over k and let g, $\tilde{g}$, α and $\tilde{\alpha}$ be the distinguished grouplike elements of A, A_R, A^* and A_R^* respectively. Then:*

(1) $g_\alpha = \tilde{g}^{-2}g$, *hence does not depend on* R.

(2) $u^2 = \tilde{g}^2$. *In particular, if* $|G(A)|$ *is odd then* $u = \tilde{g}$.

(3) *Suppose further that* A_R *is unimodular (e.g. when* A *is semisimple); then* $g_\alpha = g$, $u^2 = 1$ *and* $s^4 = I$. *If, moreover,* $|G(A)|$ *is odd then* $u = 1$ *and* $s^2 = I$.

(4) *Suppose that* $(\dim A)1 \neq 0$ *and that* $|G(A)|$ *is odd. Then* A *is semisimple and cosemisimple if and only if* A_R *is unimodular.*

(5) *Suppose that* A *is unimodular; then* $g = \tilde{g}^2$. *If, moreover,* $|G(A)|$ *is odd then* $\tilde{g} = g^{(|g|+1)/2}$.

Proof: (1) By Theorem 2.2 and Corollary 2.10, $g_{\tilde{\alpha}} = \tilde{g}^{-1}$, hence the result follows from Lemma 1.3.3.

(2) Recall that $u \in G(A_R)$. Since $\tilde{g} \in G(A_R)$ we conclude, using Corollary 2.10 and Theorem 2.2, that $u^{-1} = s(u) = u\tilde{g}^{-2}$. Since $G(A_R)$ is abelian by Theorem 2.2 this implies $(u\tilde{g}^{-1})^2 = 1$. Therefore, if $|G(A)|$ is odd then $u = \tilde{g}$.

(3) Since A_R is unimodular A_R^* is unimodular too by Theorem 2.2. Hence, $\tilde{g} = 1$ and the result follows from parts (1) and (2).

(4) Since $s^2 = I$ implies that A is semisimple and cosemisimple [LR1, Corollary 2.6] the result follows from part (3) and (7).

(5) Follows from part (1). ∎

COROLLARY 2.12: *Suppose (A, R) is a finite-dimensional triangular Hopf algebra of odd dimension over k. Then:*

(1) *If A and A^* are unimodular, then*
(2) *A_R and A_R^* are unimodular, and then*
(3) *$s^2 = I$.*

Proof: If A and A^* are unimodular then $h = 1$. But $h = \tilde{g}^{-2}$ as well. Therefore $\tilde{g}^2 = 1$. Since $|G(A)|$ is odd this implies $\tilde{g} = 1$, which is equivalent to A_R^* being unimodular. Since A_R^* is isomorphic to A_R, by Theorem 2.2, we have proved that part (1) implies part (2). Now, since $\dim A$ is odd, so is $|G(A)|$, hence by Theorem 2.11(3) part (2) implies part (3). ∎

COROLLARY 2.13: *Suppose (A, R) is an odd-dimensional triangular Hopf algebra over k. Suppose further that $(\dim A)1 \neq 0$. Then the following are equivalent:*

(1) *A and A^* are unimodular.*
(2) *A_R is unimodular.*
(3) *$s^2 = I$.*
(4) *A is semisimple.*

Proof: Now suppose $(\dim A)1 \neq 0$, and $s^2 = I$, then A and A^* are semisimple [LR1, Corollary 2.6]. This implies that A and A^* are unimodular and we are done. ∎

Example 2.14: We show that the assumption on the oddness of $|G(A)|$ in Theorem 2.2 and in Corollary 2.13 is necessary. Suppose the characteristic of k is not

2 and let $A = k\langle a, x, y | a^2 = 1, x^2 = y^2 = 0, xa = -ax, ya = -ay, xy = -yx\rangle$. Then A is a Hopf algebra where a is a grouplike element and x, y are $a:1$ skew primitive elements. Note that $s^2 \neq I$. By [G1, Propositions 2.2.1 and 2.2.3] A and A^* are unimodular of dimension 8, and (A, R) is quasitriangular if and only if

$$\begin{aligned} R = \tfrac{1}{2}\{&(1\otimes 1 + 1\otimes a + a\otimes 1 - a\otimes a) \\ &+ \alpha(x\otimes x - ax\otimes x + x\otimes ax + ax\otimes ax) \\ &+ \beta(x\otimes y + x\otimes ay - ax\otimes y + ax\otimes ay) \\ &+ \gamma(y\otimes x + y\otimes ax - ay\otimes x + ay\otimes ax) \\ &+ \delta(y\otimes y - ay\otimes y + y\otimes ay + ay\otimes ay) \\ &+ (\beta\gamma - \alpha\delta)(xy\otimes axy - axy\otimes axy + xy\otimes ay + axy\otimes xy)\} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in k$. Since $u = a(1 + (\gamma - \beta)xy)$, (A, R) is triangular if and only if $\beta = \gamma$. Note moreover that for $\beta = \gamma \neq 0$, $A_R = A$, thus A_R is unimodular while $s^2 \neq I$. If $\beta = \gamma = \delta = 0$ and $\alpha \neq 0$, A_R is Sweedler's Hopf algebra, which is self-dual and not unimodular, while A and A^* are unimodular.

Example 2.15: The assumption on the characteristic of k in Corollary 2.13 is necessary. Let k be a field of odd characteristic p. Let

$$A = k\langle e, f | [e, f] = e,\ f^p = f,\ e^p = 0\rangle$$

where e, f are primitive elements. Then, $\dim A = p^2$ is odd [LS, Section 6]. Since A is cocommutative, $s^2 = I$ and $(A, 1\otimes 1)$ is triangular. But, A and A^* are not unimodular [LS, Section 6].

3. Self-dual Hopf algebras of dimension pq^2 and their quasitriangularity

In this section we construct two families of new non-commutative non-cocommutative semisimple and cosemisimple Hopf algebras, give an explicit form of its sub-Hopf algebras and simple subcoalgebras, and study questions of quasitriangularity. These families form counterexamples of various natural questions from Section 2, and some new non-trivial ribbon unimodular Hopf algebras which can be used to compute Hennings and Kauffman's links and 3-manifolds invariants. These families are biproducts of two quasitriangular Hopf algebras $B \times H$. As will be seen in Theorems 3.11 and 3.16, quasitriangularity of

B and H will not imply quasitriangularity of $B \times H$ (obviously quasitriangularity of $B \times H$, for any B and H, implies quasitriangularity of H).

One of the main results of this paper is based on a construction of biproducts of group algebras of cyclic groups. Let $H = k[\langle\theta\rangle]$, where $\langle\theta\rangle$ is a cyclic group of order n. Assume k contains a primitive nth root of unity, η. Then H is semisimple with idempotent integral $t = (1/n)\sum_{k=0}^{n-1}\theta^k$, and $H^* = k[\langle\lambda\rangle]$, with $\langle\lambda,\theta\rangle = \eta$. Now by [CRW, 2.6] the set

$$\{\lambda^i \rightharpoonup t | i = 0, \dots, n-1\} \tag{14}$$

is a set of orthogonal idempotents of H whose sum is 1. By [CRW, 2.5.1] $\langle\lambda^k, t\rangle = \delta_{0,k}$, so we have $\langle\lambda^j, \lambda^{-i} \rightharpoonup t\rangle = \langle\lambda^{j-i}, t\rangle = \delta_{i,j}$. On the other hand, $H = (H^*)^*$, with basis P_{λ^i}, dual to the basis $\{\lambda^i\}$ of $k[\langle\lambda\rangle] = H^*$, where $\langle\lambda^j, P_{\lambda^i}\rangle = \delta_{i,j}$, hence $\lambda^{-i} \rightharpoonup t = P_{\lambda^i}$, and so

$$\Delta(\lambda^{-i} \rightharpoonup t) = \sum_{k+s=i \ (\mathrm{mod}\, n)} (\lambda^{-k} \rightharpoonup t) \otimes (\lambda^{-s} \rightharpoonup t). \tag{15}$$

Let A be an H-module algebra. The idempotents $\lambda^i \rightharpoonup t$ play an important role in defining semiinvariants, A_{λ^i}, that is

$$A_{\lambda^i} = (\lambda^{-i} \rightharpoonup t) \cdot A = \{a \in A | h \cdot a = \langle\lambda^i, h\rangle a, \text{ all } h \in H\}.$$

In particular, suppose G is any finite group and $\theta: G \to G$ is an automorphism of order n. Extend θ to $A = k[G]$ linearly, thus making A an $H = k[\langle\theta\rangle]$-module algebra. Moreover, A is an H-module coalgebra, since both A and H are group algebras. Thus, by (15), if $a \in G$, then

$$\Delta((\lambda^{-i} \rightharpoonup t) \cdot a) = \sum_{k+s=i \ (\mathrm{mod}\, n)} (\lambda^{-k} \rightharpoonup t) \cdot a \otimes (\lambda^{-s} \rightharpoonup t) \cdot a. \tag{16}$$

By (14),

$$A = \bigoplus_{i=0}^{n-1} (\lambda^{-i} \rightharpoonup t) \cdot A = \bigoplus_{i=0}^{n-1} A_{\lambda^i}. \tag{17}$$

Now let $\{c_j\}_{j=0}^r$, with $c_0 = 1$ and $c_1 = b$, be a set of representatives of the disjoint orbits of the action of θ on G; then by (14) and (17) the non-zero elements in $\{(\lambda^{-i} \rightharpoonup t) \cdot c_j | 0 \le j \le r, \ \ 0 \le i \le n-1\}$ form a k-linearly independent set. It furthermore spans A, for if x belongs to the same orbit as c_j, that is, $x = \theta^k(c_j)$, then by [CRW, 2.1.1]

$$(\lambda^{-i} \rightharpoonup t) \cdot x = (\lambda^{-i} \rightharpoonup t) \cdot \theta^k(c_j) = \langle\lambda^i, \theta^k\rangle(\lambda^{-i} \rightharpoonup t) \cdot c_j. \tag{18}$$

Since $x = \sum_{i=0}^{n-1}(\lambda^{-i} \rightharpoonup t) \cdot x$, we are done.

Set $b_i^j = (\lambda^{-i} \rightharpoonup t) \cdot c_j$, then by (16)

$$\Delta(b_i^j) = \sum_{t+r=i (\mathrm{mod}\, n)} b_t^j \otimes b_r^j. \tag{19}$$

Note also that

$$\varepsilon(b_i^j) = \delta_{i,0}. \tag{20}$$

Suppose now that $G = \langle b \rangle$ is a cyclic group and that m is in the group of units of the integers modulo $|b|$. Let $\langle h \rangle$ be a second finite cyclic group such that $|m|$ divides $|h|$, and set $n = |h|/|m|$. Using the above we construct now the biproduct $k[\langle b \rangle] \times k[\langle h \rangle]$. Define τ: $\langle h \rangle \to \mathrm{Aut}(\langle b \rangle)$ via $h^i \cdot b^j = b^{jm^i}$. Then, τ is a well defined group homomorphism by the above assumptions and $\ker \tau = \langle a \rangle$ is of order n, where $a = h^{|m|}$. Moreover, $k[\langle b \rangle]$ becomes a left module algebra over $k[\langle h \rangle]$ via extending τ linearly. Let us denote (with abuse of notation) this action by τ: $k[\langle h \rangle] \otimes k[\langle b \rangle] \to k[\langle b \rangle]$. Suppose there exists θ: $\langle b \rangle \to \langle b \rangle$, an automorphism of order n. Extend θ to $k[\langle b \rangle]$ linearly. Suppose further that the field k contains primitives $|h|$th and $|b|$th roots of unity, and let η be a primitive nth root of unity. Set $k[\langle b \rangle]_i = k[\langle b \rangle]_{\lambda^i}$, and define ρ: $k[\langle b \rangle] \to k[\langle h \rangle] \otimes k[\langle b \rangle]$ via $\rho(b_i^j) = a^i \otimes b_i^j$. Then it is straightforward to check that $(k[\langle b \rangle], k[\langle h \rangle])$ is an admissible pair with structure maps τ and ρ indicated above. Hence the biproduct $A = k[\langle b \rangle] \times k[\langle h \rangle]$ is a well defined Hopf algebra. Observe that as an algebra, A is the group algebra of the semidirect product group of $\langle b \rangle$ and $\langle h \rangle$.

If moreover $|h| = n^2$, we may choose $\theta = \tau(h)$ and then much more can be said. Let $\gamma \in k[\langle h \rangle]^*$ be so that $\langle \gamma, h \rangle$ is a primitive $|h|$th root of unity with $\langle \gamma, h \rangle^n = \eta$, and let $\beta \in k[\langle b \rangle]^*$ be so that $\langle \beta, b \rangle$ is a primitive $|b|$th root of unity. Then $k[\langle b \rangle]^* = k[\langle \beta \rangle]$ and $k[\langle h \rangle]^* = k[\langle \gamma \rangle]$. Let f: $k[\langle \beta \rangle] \to k[\langle b \rangle]$ and g: $k[\langle \gamma \rangle] \to k[\langle h \rangle]$ be the Hopf algebra isomorphisms determined by $f(\beta) = b$ and $g(\gamma) = h$. Observe that by Theorem 1.2.1, $A^* = k[\langle b \rangle]^* \times k[\langle h \rangle]^*$ with structure maps τ^* and ρ^*. Also, replacing θ by θ^* yields a linear basis $\{\beta_i^j\}$ of $k[\langle \beta \rangle]$.

PROPOSITION 3.1: *Let $|h| = n^2$, and suppose that the field k contains primitives n^2th and $|b|$th roots of unity. Let $A = k[\langle b \rangle] \times k[\langle h \rangle]$, f, g, τ and η be as above with $\theta = \tau(h)$. Then:*

(1) *$f \times g$: $A^* \to A$ is an isomorphism of Hopf algebras.*

(2) *$\langle \beta_k^l, b_i^j \rangle = 0$ if $i \neq k$.*

Proof: (1) First note that $\eta^i = \langle\gamma^{ni}, h\rangle = \langle\gamma^i, h^n\rangle = \langle\gamma^i, a\rangle$, where the last equality follows since $a = h^n$. Denoting by $\cdot$ the action induced by ρ^* we have

$$\begin{aligned}\langle\gamma\cdot\beta, b_i^j\rangle &= \langle\rho^*(\gamma\otimes\beta), b_i^j\rangle\\ &= \langle\gamma\otimes\beta, a^i\otimes b_i^j\rangle = \eta^i\langle\beta, b_i^j\rangle\\ &= \langle\beta, \theta(b_i^j)\rangle = \langle\theta^*(\beta), b_i^j\rangle\end{aligned}$$

for all $b_i^j\in k[\langle b\rangle]_i$. Thus, $\gamma\cdot\beta = \theta^*(\beta)$ so γ^n generates the kernel of this action, and it is easy to verify that $\theta^*(\beta) = \beta^m$.

Repeating the above construction by replacing θ by θ^* yields a linear basis $\{\beta_i^j\}$ of $k[\langle\beta\rangle]$, with $\tau^*(\beta_i^j) = \gamma^{ni}\otimes\beta_i^j$. Indeed,

$$\begin{aligned}\langle\tau^*(\beta_i^j), h\otimes b_s^l\rangle &= \langle\beta_i^j, h\cdot b_s^l\rangle\\ &= \langle\beta_i^j, \theta(b_s^l)\rangle = \langle\theta^*(\beta_i^j), b_s^l\rangle\\ &= \eta^i\langle\beta_i^j, b_s^l\rangle = \langle\gamma^{ni}\otimes\beta_i^j, h\otimes b_s^l\rangle\end{aligned}$$

for all $b_s^l\in k[\langle b\rangle]_s$, thus $\tau^*(\beta_i^j) = \gamma^{ni}\otimes\beta_i^j$.

Now, by Theorem 1.2.3 it is sufficient to show that

$$f(\gamma\cdot\beta) = g(\gamma)\cdot f(\beta) \quad\text{and}\quad \rho(f(\beta_i^j)) = (g\otimes f)\tau^*(\beta_i^j).$$

Indeed,

$$f(\gamma\cdot\beta) = f(\beta^m) = b^m = h\cdot b = g(\gamma)\cdot f(\beta).$$

For any θ: $k[\langle b\rangle]\to k[\langle b\rangle]$

$$\langle\theta(f(\beta)),\beta\rangle = \langle\theta(b),\beta\rangle = \langle b,\theta^*(\beta)\rangle = \langle f(\beta),\theta^*(\beta)\rangle = \langle\beta, f^*(\theta^*(\beta))\rangle.$$

Setting $\theta = \mathrm{id}$, we have shown that $f = f^*$, and hence by the above $\theta f = f\theta^*$ for all such θ. Therefore $\theta(f(\beta_i^j)) = f(\theta^*(\beta_i^j)) = \eta^i f(\beta_i^j)$ and we have

$$\begin{aligned}\rho(f(\beta_i^j)) &= a^i\otimes f(\beta_i^j) = h^{ni}\otimes f(\beta_i^j)\\ &= (g\otimes f)(\gamma^{ni}\otimes\beta_i^j) = (g\otimes f)\tau^*(\beta_i^j).\end{aligned}$$

(2) Since

$$\begin{aligned}\langle\beta_k^l, b_i^j\rangle &= \eta^{-i}\langle\beta_k^l, \eta^i b_i^j\rangle = \eta^{-i}\langle\beta_k^l, \theta(b_i^j)\rangle\\ &= \eta^{-i}\langle\theta^*(\beta_k^l), b_i^j\rangle = \eta^{k-i}\langle\beta_k^l, b_i^j\rangle\end{aligned}$$

the result follows. This completes the proof of the proposition. ∎

Remark 3.2: Since $\{b_i^j\}$ forms a linear basis for $k[\langle b\rangle]$ and $\beta_i^l \neq 0$, it follows from part (2) of the above proposition that for any l there exists j such that $\langle \beta_i^l, b_i^j\rangle \neq 0$.

We now specialize: Let p and q be prime numbers satisfying $p = 1 \pmod q$, and let $m \in Z_p$ so that $|m| = q$. Let $\langle b\rangle$ be of order p, and $\langle h\rangle$ of order q^2. From now on we shall assume, unless otherwise stated, that the field k contains primitive pth and q^2th roots of unity. As above let $\theta(b) = h \cdot b = b^m$. Denote by A_{qp} the resulting Hopf algebra. Note that $\langle h^q\rangle$ is the unique subgroup of $\langle h\rangle$ of order q, and that h^q acts trivially, hence $k[\langle h^q\rangle] \subset Z(A_{qp})$. Moreover, by Proposition 3.1, A_{qp} is self-dual. Using the linear basis

$$\{b_i^j | 0 \le i \le q-1,\ 0 \le j \le (p-1)/q\}$$

of A and (19) we have that

$$(b_i^j \times h^s)(b_k^l \times h^t) = \eta^{sk}(b_i^j b_k^l \times h^{s+t}) \tag{21}$$

and

$$\Delta(b_i^j \times h^s) = \sum_{t=0}^{q-1} b_{i-t}^j \times h^{qt+s} \otimes b_t^j \times h^s. \tag{22}$$

By (4) we have that

$$s(b_i^j \times h^t) = \eta^{-ti}(s(b_i^j) \times h^{-qi-t}). \tag{23}$$

In the following proposition we describe the coradical of A_{qp} explicitly and show that A_{qp} is cosemisimple, and hence also semisimple. When there is no danger of ambiguity we identify h^i with $1 \times h^i$.

PROPOSITION 3.3: *Let $A = A_{qp}$ and k be as before. Then:*

(1) *$G(A) = \langle h\rangle$.*

(2) *A has $p-1$ simple subcoalgebras of dimension q^2.*

(3) *A is cosemisimple and semisimple.*

Proof: By (21) and (22), $\langle h\rangle$ is a subgroup of $G(A)$. Thus, q^2 divides $|G(A)|$. Since A is not cocommutative (see (22)), $A \neq k[G(A)]$ and thus $|G(A)| < pq^2$, but $|G(A)|$ divides $\dim A = pq^2$ thus $|G(A)| = q^2$ and hence $G(A) = \langle h\rangle$.

Since by (22), $\Delta(b_i^j \times 1) = \sum_{r+s=i}(b_r^j \times h^{qs} \otimes b_s^j \times 1)$, it follows that $I = \mathrm{sp}_k\{b_i^j \times 1 | 0 \le i \le q-1\}$ is a left coideal of A of dimension q. We show that I is a simple left coideal of A by showing it is a simple right A^*-module. Indeed, let $0 \neq a = \sum_{i=0}^{q-1} \alpha_i (b_i^j \times 1) \in I$, and suppose that $\alpha_l \neq 0$ for some l. Let $P_t = (b_{l-t}^j)^* \in k[\langle b \rangle]^*$ for any $0 \leq t \leq q-1$. Then, for any such t, $a \leftharpoonup (P_t \times (h^{qt})^*) = \sum_{i=0}^{q-1} \alpha_i \sum_{s=0}^{q-1} \langle P_t, b_{i-s}^j \rangle \langle (h^{qt})^*, h^{qs} \rangle b_s^j \times 1 = \alpha_l (b_t^j \times 1)$, and hence I is a simple right A^*-module.

By [L, page 354], since I is a simple left A-coideal, $A_0^k = L(I) = \mathrm{sp}_k\{b_i^k \times h^{qj} | 0 \leq i,j \leq q-1\}$ for $1 \leq k \leq (p-1)/q$ is a simple sub-coalgebra of A, of dimension q^2, for all k. Now, set $A_n^k = \mathrm{sp}_k\{b_i^k \times h^{qj+n} | 0 \leq i,j \leq q-1\}$ for $1 \leq k \leq (p-1)/q$ and $0 \leq n \leq q-1$. Since $A_n^k = A_0^k h^n$, $h^n \in G(A)$ and multiplication by a grouplike element is an isomorphism of coalgebras, it follows that A_n^k is a simple sub-coalgebra of A for all k and n. Since $\dim A_n^k = q^2$, and since there are $q(p-1)/q = p-1$ such simple sub-coalgebras, their dimensions sum to $(p-1)q^2$. Finally, since there are q^2 grouplike elements and $q^2 + (p-1)q^2 = pq^2 = \dim A$, the result follows. ∎

Remark 3.4: Note that the elements of A_n^k commute if and only if $n = 0$.

In the following proposition we describe the sub-Hopf algebras of A_{qp}.

PROPOSITION 3.5: *Let $A = A_{qp}$ and k be as before. Then the non-trivial sub-Hopf algebras of A are $k[\langle h^q \rangle]$, $k[\langle h \rangle]$ and*

$$B = \mathrm{sp}_k\{b^i \times h^{qj} | 0 \leq i \leq p-1,\ 0 \leq j \leq q-1\}$$

of dimensions q, q^2 and pq, respectively.

Proof: Let B be a non-trivial sub-Hopf algebra of A. By [NZ], $\dim B = p$, q, q^2 or pq. If $\dim B = p$ then $B = k[G(B)]$ by [Z]. But, by Proposition 3.3, $|G(B)|$ must divide $|G(A)| = q^2$ and hence this is impossible. If $\dim B = q$ then $B = k[G(B)]$ by [Z], and the only possibility is $B = k[\langle h^q \rangle]$. Suppose $\dim B = q^2$. By Proposition 3.3, B is cosemisimple and the dimension of the simple sub-coalgebras is q^2 or 1. Since B contains 1, a grouplike element, it follows that $B = k[\langle h \rangle]$. Suppose now that $\dim B = pq$, then $G(B) = k[\langle h^q \rangle]$. In particular this implies that the image of the restriction to B, of the projection π: $A \to k[\langle h \rangle]$, is a sub-Hopf algebra of $k[\langle h \rangle]$ containing $k[\langle h^q \rangle]$. Therefore, $\mathrm{Im}(\pi_{|B}) = k[\langle h^q \rangle]$ or $k[\langle h \rangle]$. Dualizing implies that $[\mathrm{Im}(\pi_{|B})]^*$ is embedded in B^*, hence $\dim[\mathrm{Im}(\pi_{|B})]^*$ divides pq, hence must be q and hence $\mathrm{Im}(\pi_{|B}) = k[\langle h^q \rangle]$. Thus, we have the following sequence of maps: $k[\langle h^q \rangle] \xrightarrow{i} B \xrightarrow{\pi} k[\langle h^q \rangle]$. By [R1, Theorem 3], $B = B' \times k[\langle h^q \rangle]$ for some B'. Since $k[\langle h^q \rangle]$ acts on B' via ad_i [R1, 3.4], and $k[\langle h^q \rangle] \subset Z(A)$, it follows that the action is trivial. But then it follows from [R1, 2.8(a)] that B' is a sub-Hopf algebra of A of dimension p. Therefore by [Z], $B' \cong k[Z_p]$, hence commutative. This implies that B is commutative. Therefore B is a direct sum of q 1-dimensional and $(p-1)/q$ q^2-dimensional commutative simple sub-coalgebras of A. Using Proposition 3.3 and Remark 3.4 we conclude that $B = k[\langle h^q \rangle] \oplus (\bigoplus_{k=1}^{(p-1)/q} A_0^k)$. This completes the proof of the proposition. ∎

Let us single out some of the properties proved above.

Remark 3.6: Let B be the unique pq-dimensional sub-Hopf algebra described in Proposition 3.5. Then $B = k[\langle b \rangle] \times k[\langle h^q \rangle]$, with the trivial action, and the coaction induced by ρ. Thus, as an algebra, $B = k[\langle b \rangle \times \langle h^q \rangle]$ is the group algebra of the commutative group $\langle b \rangle \times \langle h^q \rangle$. Now, let β and α be so that $k[\langle b \rangle]^* = k[\langle \beta \rangle]$ and $k[\langle h^q \rangle]^* = k[\langle \alpha \rangle]$. We wish to show that $B^* = k[\langle \beta \rangle \rtimes \langle \alpha \rangle]$ is the group algebra, as a Hopf algebra, of the semidirect product group of $\langle \beta \rangle$ and $\langle \alpha \rangle$, where $\alpha \cdot \beta = \beta^m$. Indeed, by 3.1, $B^* = k[\langle \beta \rangle] \times k[\langle \alpha \rangle]$ with the trivial coaction, and action determined by $\alpha \cdot \beta = \theta^*(\beta) = \beta^m$. In particular, $B^* = k[\langle \beta \rangle] \otimes k[\langle \alpha \rangle]$ as a coalgebra, and $B^* = k[\langle \beta \rangle \rtimes \langle \alpha \rangle]$ as an algebra. Since any element of $\langle \beta \rangle \rtimes \langle \alpha \rangle$ is a grouplike element we are done.

It is this B that enables us to prove the following lemma which is basic in our analysis of quasitriangularity of $A = A_{qp}$. Among the rest we prove quasitriangularity by exhibiting an isomorphism of A and A^{cop}, which by 3.1 is an isomorphism of A and A^{*cop}, and continue by showing that then (QT.5)′ is satisfied. All this is possible if and only if $q = 2$.

LEMMA 3.7: *Let $A = A_{qp}$ and k be as before. Then:*

(1) *$A \cong A^{cop}$ if and only if $q = 2$. Furthermore, if $f\colon A_{2p} \to A_{2p}^{cop}$ is an isomorphism of Hopf algebras, then f is determined by*

$$f(h) = h^{2i+3} \quad \text{and} \quad f(b \times 1) = s(b^r \times 1)$$

for some $0 \le i \le 1$ and $1 \le r \le p-1$.

(2) *Let f be an automorphism of A. Then f is determined by*

$$f(h) = h^{qi+1} \quad \text{and} \quad f(b \times 1) = b^r \times 1$$

for some $0 \le i \le q-1$ and $1 \le r \le p-1$. Furthermore, $\mathrm{Aut}_{\mathrm{Hopf}}(A) \cong Z_q \times Z_p^{\times}$.

Proof: We first determine $\mathrm{Aut}_{\mathrm{Hopf}}(B)$ by determining $\mathrm{Aut}(k[\langle\beta\rangle] \times k[\langle\alpha\rangle])$. The latter is determined by the automorphisms of the semidirect product group of $\langle\beta\rangle$ and $\langle\alpha\rangle$. These automorphisms can be determined as follows: Let $G = \langle\beta\rangle \rtimes \langle\alpha\rangle$. Then by Sylow's Theorem, $\langle\beta\rangle$ is the unique normal Sylow p-subgroup of G. Set $G_j = \{\beta^{j(m^i-1)} \rtimes \alpha^i | 0 \le i \le q-1\}$ for $0 \le j \le p-1$. Then each G_j is a Sylow q-subgroup of G, and by Sylow's Theorem, $\{G_j | 0 \le j \le p-1\}$ is the set of all Sylow q-subgroups of G. Let ϕ be an automorphism of G. Since $\phi(\langle\beta\rangle) = \langle\beta\rangle$ and $\phi(G_j) = G_{j'}$ it is not hard to verify that

$$\phi(\beta^l \rtimes \alpha^k) = \beta^{rl+j(m^{ik}-1)} \rtimes \alpha^{ik}$$

for some $1 \le i \le q-1$, $0 \le j \le p-1$ and $1 \le r \le p-1$. Since $(1 \rtimes \alpha)(\beta \rtimes 1)(1 \rtimes \alpha)^{-1} = \beta^m \rtimes 1$ it follows that $i = 1$ and hence that

$$\phi(\beta^l \rtimes \alpha^k) = \beta^{rl+j(m^k-1)} \rtimes \alpha^k$$

for some $0 \le j \le p-1$ and $1 \le r \le p-1$. Finally, it is not hard to verify that ϕ described in the last equality determines a well defined automorphism, hence determines an automorphism of $k[G] = B^*$. Dualizing yields an automorphism $\phi^*: B \to B$ which is determined by

$$\phi^*(1 \times h^q) = 1 \times h^q \quad \text{and} \quad \phi^*(b \times 1) = b^r \times \left(\sum_{v=0}^{q-1} \alpha_{v,j} h^{qv}\right) \tag{24}$$

where $\alpha_{v,j} = \langle\beta, b\rangle^{-j} \langle\beta^j, b_v^1\rangle$, $0 \le j \le p-1$ and $1 \le r \le p-1$. Since $\langle\alpha, h^q\rangle = \eta$, the second equality follows by

$$\begin{aligned}
\langle\phi^*(b \times 1), \beta^l \times \alpha^k\rangle &= \langle b \times 1, \beta^{rl+j(m^k-1)} \times \alpha^k\rangle \\
&= \langle b, \beta^{rl+j(m^k-1)}\rangle \\
&= \langle b, \beta^{rl}\rangle\langle b, \beta^{j(m^k-1)}\rangle \\
&= \langle b^r, \beta^l\rangle\langle b, \beta^{jm^k}\rangle\langle b, \beta^{-j}\rangle \\
&= \langle b^r, \beta^l\rangle\langle b, \theta^{*k}(\beta^j)\rangle\langle b, \beta^{-j}\rangle \\
&= \langle b^r, \beta^l\rangle\langle\theta^k(b), \beta^j\rangle\langle b, \beta\rangle^{-j}
\end{aligned}$$

$$= \langle b^r, \beta^l\rangle\langle\theta^k(\sum_{v=0}^{q-1}(\lambda^{-v} \rightharpoonup t)\cdot b), \beta^j\rangle\langle b,\beta\rangle^{-j}$$

$$= \langle b^r, \beta^l\rangle\langle\sum_{v=0}^{q-1}\eta^{vk}b_v^1, \beta^j\rangle\langle b,\beta\rangle^{-j}$$

$$= \langle b^r, \beta^l\rangle\sum_{v=0}^{q-1}\langle h^{qv}, \alpha^k\rangle\langle b_v^1, \beta^j\rangle\langle b,\beta\rangle^{-j}$$

$$= \langle b^r, \beta^l\rangle\langle\sum_{v=0}^{q-1}\langle b_v^1, \beta^j\rangle\langle b,\beta\rangle^{-j}h^{qv}, \alpha^k\rangle$$

$$= \langle b^r \times (\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}), \beta^l \times \alpha^k\rangle.$$

(1) Suppose now that $f\colon A \to A^{cop}$ is an isomorphism of Hopf algebras. Then, $s \circ f\colon A \to A^{op}$ is an isomorphism of Hopf algebras, and since B is the unique (commutative) sub-Hopf algebra of A and A^{op}, of dimension pq, $(s \circ f)_{|B}$ is an automorphism of B. Therefore, since $(s \circ f)(h^q) = h^q$ we have that $(s \circ f)(h) = h^{qi+1}$ for some $0 \le i \le q-1$, and since $s^2 = I$ it follows from (24) that $f = s\circ(s\circ f)$ must be of the form

$$f(1 \times h) = 1 \times h^{-qi-1} \quad \text{and} \quad f(b \times 1) = s\left(b^r \times \sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right) \tag{25}$$

where $\alpha_{v,j} = \langle\beta, b\rangle^{-j}\langle\beta^j, b_v^1\rangle$, $0 \le j \le p-1$, $1 \le r \le p-1$ and $0 \le i \le q-1$.

Since on one hand

$$\begin{aligned} f\left[(1 \times h)(b \times 1)\right] &= f(h \cdot b \times h) \\ &= f(b^m \times h) = f(b \times 1)^m f(1 \times h) \\ &= s\left(b^{mr} \times \left(\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right)^m\right)\left(1 \times h^{-qi-1}\right) \\ &= s\left[\left(1 \times h^{qi+1}\right)\left(b^{mr} \times \left(\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right)^m\right)\right] \\ &= s\left(b^{m^2 r} \times h^{qi+1}\left(\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right)^m\right) \end{aligned}$$

and on the other hand

$$f(1\times h)f(b\times 1)=(1\times h^{-qi-1})s\left(b^r\times\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right)$$
$$=s\left(b^r\times h^{qi+1}\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\right)$$

it follows that if f is an isomorphism then

$$b^{m^2r}\langle\alpha^i,\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\rangle^m=b^r\langle\alpha^i,\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\rangle \tag{26}$$

for all $0\le i\le q-1$. We show that $\langle\alpha^i,\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\rangle=\langle\beta,b\rangle^{j(m^i-1)}$ for all $0\le i\le q-1$. This will imply that if f is an isomorphism then $j=0$, and $b^{m^2r}=b^r$, and hence that $m^2=1(\operatorname{mod} q)$. Since q is the order of m in Z_p it will follow that $q=2$. Indeed, by (14)

$$\begin{aligned}\langle\alpha^i,\sum_{v=0}^{q-1}\alpha_{v,j}h^{qv}\rangle&=\sum_{v=0}^{q-1}\alpha_{v,j}\eta^{iv}\\&=\langle\beta,b\rangle^{-j}\sum_{v=0}^{q-1}\langle\beta^j,b_v^1\rangle\eta^{iv}\\&=\langle\beta,b\rangle^{-j}\langle\beta^j,\sum_{v=0}^{q-1}\eta^{iv}b_v^1\rangle\\&=\langle\beta,b\rangle^{-j}\langle\beta^j,\theta^i(\sum_{v=0}^{q-1}b_v^1)\rangle\\&=\langle\beta,b\rangle^{-j}\langle\theta^{*i}(\beta^j),\sum_{v=0}^{q-1}(\lambda^{-v}\rightharpoonup t)\cdot b\rangle\\&=\langle\beta,b\rangle^{-j}\langle\beta^{jm^i},b\rangle\\&=\langle\beta,b\rangle^{j(m^i-1)}.\end{aligned}$$

Thus, (26) is equivalent to $\langle\beta,b\rangle^{mj(m^i-1)}b^{m^2r}=\langle\beta,b\rangle^{j(m^i-1)}b^r$ and the result follows.

Conversely, if $q=2$ then, by the above, if

$$(\sum_{v=0}^{1}\alpha_{v,j}h^{2v})^m=\sum_{v=0}^{1}\alpha_{v,j}h^{2v},$$

then f, given in (25), is an isomorphism. Now, if $j = 0$, then $\alpha_{v,0} = \varepsilon(b_v^1) = \delta_{v,0}$, and hence $(\sum_{v=0}^{1} \alpha_{v,0}h^{2v})^m = 1 = \sum_{v=0}^{1} \alpha_{v,0}h^{2v}$. Therefore, we conclude that the map $f\colon A_{2p} \to A_{2p}^{cop}$ determined by

$$f(h) = h^{2i+3} \quad \text{and} \quad f(b \times 1) = s(b^r \times 1)$$

is an isomorphism of Hopf algebras for all $1 \le r \le p-1$ and $0 \le i \le 1$. This completes the proof of the lemma.

(2) Let $f\colon A \to A$ be an automorphism of A. Then using similar arguments to those used in the proof of part (1) yields that f must be of the form

$$f(1 \times h) = 1 \times h^{qi+1} \quad \text{and} \quad f(b \times 1) = b^r \times \sum_{v=0}^{q-1} \alpha_{v,j} h^{qv}$$

where $\alpha_{v,j} = \langle \beta, b\rangle^{-j} \langle \beta^j, b_v^1\rangle$, $0 \le j \le p-1$, $1 \le r \le p-1$ and $0 \le i \le q-1$. Since $f((1 \times h)(b \times 1)) = f(1 \times h)f(b \times 1)$ it follows, as before, that $j = 0$, and hence that f must be determined by

$$f(1 \times h) = 1 \times h^{qi+1} \quad \text{and} \quad f(b \times 1) = b^r \times 1$$

where $1 \le r \le p-1$ and $0 \le i \le q-1$. It is not hard to verify that f described in the last equality determines an automorphism of A. Denote such f by $f_{i,r}$. Since $f_{i,r} \circ f_{j,t} = f_{i+j,rt}$ where the sum $i + j$ is mod q and the multiplication rt is mod p, the result follows. ∎

Using Lemma 3.7 we determine first when A_{2p} is minimal quasitriangular:

THEOREM 3.8: *Suppose that the field k contains primitive 4th and pth roots of unity and let $A = A_{2p}$. Then the maps $f, f'\colon A^{*cop} \to A$ given by $f(\beta_i^k \times \gamma^j) = b_i^{rk} \times h^{2i+j}$ and $f'(\beta_i^k \times \gamma^j) = b_i^{rk} \times h^{2i-j}$ determine two minimal quasitriangular structures on A for any $1 \le r \le p-1$.*

Proof: We first show that f and f' are Hopf algebra maps. Since A is isomorphic to A^* by Proposition 3.1, it is sufficient to show that the maps $b_i^k \times h^j \mapsto b_i^{rk} \times h^{2i+j}$ and $b_i^k \times h^j \mapsto b_i^{rk} \times h^{2i-j}$ are Hopf algebra maps. Indeed, the first map is obtained from (25) after substituting $i = 1$, while the second one is obtained from (25) after substituting $i = 0$. It remains to check that f and f' satisfy (QT.5)$'$. Note that since $q = 2$, $b_0^j = \frac{1}{2}(b^j + b^{-j})$ and $\beta_0^j = \frac{1}{2}(\beta^j + \beta^{-j})$ for $0 \le j \le (p-1)/2$, and $b_1^j = \frac{1}{2}(b^j - b^{-j})$ and $\beta_1^j = \frac{1}{2}(\beta^j - \beta^{-j})$ for $1 \le j \le (p-1)/2$.

Since A^* and A are generated as algebras by $G = \{\beta_0^j \times \varepsilon, \beta_1^j \times \varepsilon, \varepsilon \times \gamma\}$ and $G' = \{b_0^j \times 1, b_1^j \times 1, 1 \times h\}$ respectively, it is sufficient to check (QT.5)$'$ for $p \in G$ and $a \in G'$. We check it, for example, for f, $p = \beta_1^j \times \varepsilon$ and $a = b_1^m \times 1$. By (22), $\Delta(\beta_1^j \times \varepsilon) = \beta_1^j \times \varepsilon \otimes \beta_0^j \times \varepsilon + \beta_0^j \times \gamma^2 \otimes \beta_1^j \times \varepsilon$ and $\Delta(b_1^m \times 1) = b_1^m \times 1 \otimes b_0^j \times 1 + b_0^j \times h^2 \otimes b_1^j \times 1$. Therefore, using Proposition 3.1(2) we compute

$$\begin{aligned}
&\sum \langle p_{(1)}, a_{(2)} \rangle a_{(1)} f(p_{(2)}) \\
&\quad = \langle \beta_1^j, b_1^m \rangle (b_0^m \times h^2) f(\beta_0^j \times \varepsilon) + \langle \beta_0^j, b_0^m \rangle (b_1^m \times 1) f(\beta_1^j \times \varepsilon) \\
&\quad = \langle \beta_1^j, b_1^m \rangle b_0^m b_0^{jr} \times h^2 + \langle \beta_0^j, b_0^m \rangle b_1^m b_1^{jr} \times h^2
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
&\sum \langle p_{(2)}, a_{(1)} \rangle f(p_{(1)}) a_{(2)} \\
&\quad = \langle \beta_0^j, b_0^m \rangle f(\beta_1^j \times \varepsilon)(b_1^m \times 1) + \langle \beta_1^j, b_1^m \rangle f(\beta_0^j \times \alpha^2)(b_0^m \times 1) \\
&\quad = \langle \beta_0^j, b_0^m \rangle b_1^m b_1^{jr} \times h^2 + \langle \beta_1^j, b_1^m \rangle b_0^m b_0^{jr} \times h^2.
\end{aligned}$$

Finally, since f is an isomorphism, $\mathrm{Im}(f) = A$ and hence f determines a minimal quasitriangular structure on A. This completes the proof of the theorem. ∎

Having established when $A = A_{qp}$ is minimal quasitriangular we now go on to study when (A, R) is quasitriangular but not minimal. This would imply that A_R is a proper sub-Hopf algebra of A. By Proposition 3.5 and the fact that B is commutative but not cocommutative, the only possibility for A_R is to be a sub-Hopf algebra of $k[G(A)]$, hence $R \in k[G(A)] \otimes k[G(A)]$. We show:

Lemma 3.9: *Let $A = A_{qp}$ and k be as before. Then, there exists $R \in k[G(A)] \otimes k[G(A)]$ such that (A, R) is quasitriangular if and only if $q = 2$. Moreover, A_{2p} admits exactly two such structures none of which is triangular.*

Proof: Assume such R exists and let $f = f_R$. By Proposition 3.3, $\mathrm{Im}(f)$ is a sub-Hopf algebra of $k[\langle h \rangle]$, and hence $R \in k[\langle h \rangle] \otimes k[\langle h \rangle]$. Let $\omega \in k$ be a primitive q^2th root of unity. By [R2, page 219], there exists $0 \le n \le q^2 - 1$ so that $R = R_n$, where

$$R_n = \sum_{l,k=0}^{q^2-1} \frac{\omega^{-lk}}{q^2} \left(1 \times h^l\right) \otimes \left(1 \times h^{nk}\right).$$

Observe that $f_n = f_{R_n}\colon A^{*cop} \to A$ is given by $f_n(\beta_i^j \times \gamma^k) = \varepsilon(b_i^j)(1 \times h^{nk})$. We now show that f_n satisfies (QT.5)$'$ if and only if $q = 2$ and $n = 1, 3$. Let

$p=\beta_i^m\times\gamma^j$ and $a=b_t^l\times h^r$. Then, on one hand

$$\begin{aligned}\sum\langle p_{(1)},a_{(2)}\rangle a_{(1)}f_n(p_{(2)})\\ &=\sum_{u,v=0}^{q-1}\langle\beta_{i-u}^m,b_v^l\rangle\langle\gamma^{qu+j},h^r\rangle(b_{t-v}^l\times h^{qv+r})\varepsilon(\beta_u^m)(1\times h^{nj})\\ &=\sum_{v=0}^{q-1}\langle\beta_i^m,b_v^l\rangle\langle\gamma,h\rangle^{rj}(b_{t-v}^l\times h^{qv+r+nj})\varepsilon(\beta_0^m)\\ &=\langle\beta_i^m,b_i^l\rangle\langle\gamma,h\rangle^{rj}(b_{t-i}^l\times h^{qi+r+nj})\varepsilon(\beta_0^m)\end{aligned}$$

and on the other hand

$$\begin{aligned}\sum\langle p_{(2)},a_{(1)}\rangle f_n(p_{(1)})a_{(2)}\\ &=\sum_{u,v=0}^{q-1}\langle\beta_u^m,b_{t-v}^l\rangle\langle\gamma^j,h^{qv+r}\rangle(1\times h^{nqu+nj})(b_v^l\times h^r)\varepsilon(\beta_{i-u}^m)\\ &=\sum_{v=0}^{q-1}\langle\beta_i^m,b_{t-v}^l\rangle\langle\gamma,h\rangle^{j(qv+r)}(b_v^l\times h^{nqi+nj+r})\varepsilon(\beta_0^m)\eta^{jvn}\\ &=\langle\beta_i^m,b_i^l\rangle\langle\gamma,h\rangle^{j(qt-qi+r)}(b_{t-i}^l\times h^{nqi+r+jn})\varepsilon(\beta_0^m)\eta^{j(t-i)n}.\end{aligned}$$

Since $\langle\gamma,h\rangle^q=\eta$ and $\varepsilon(\beta_0^m)=1$, an equality holds if and only if

$$\langle\beta_i^m,b_i^l\rangle h^{qi}=\langle\beta_i^m,b_i^l\rangle\eta^{j(n+1)(t-i)}h^{nqi}\tag{27}$$

for all i,m,l and t. By Remark 3.2, there exists l such that $\langle\beta_1^1,b_1^l\rangle\neq0$. Thus, if $i=m=1$ then (27) holds if and only if $h^q=\eta^{j(n+1)(t-1)}h^{nq}$. Since the order of h is q^2, $q=nq(\operatorname{mod}q^2)$, hence $n=n'q+1$ for some n', and thus since $\eta^q=1$ we have $1=\eta^{j(n+1)(t-1)}=\eta^{2j(t-1)}$ for all j,t. This holds if and only if $\eta^2=1$ and $n=n'q+1$, i.e. if and only if $q=2$ and $n=1$ or 3. It is not hard to verify that if $q=2$ and $n=1$ or 3 then (27) holds for all i,m,l and t. This completes the proof of the lemma. ∎

COROLLARY 3.10: *By Theorem 2.3, A_{2p} is not minimal triangular, since its group of grouplike elements forms a cyclic group of order 4, and $k[G(A_{2p})]\cong k[Z_4]$ does not admit minimal triangular structures. Thus, A_{2p} is not triangular by Theorem 3.8 and Lemma 3.9.*

We now summarize:

THEOREM 3.11: *Let p and q be prime numbers satisfying $p = 1(\mathrm{mod}\, q)$, and let k be a field containing primitive pth and q^2th roots of unity. Then A_{qp} is a self-dual semisimple Hopf algebra of dimension pq^2, and A_{qp} is quasitriangular if and only if $q = 2$. Furthermore, A_{2p} admits exactly $2p-2$ minimal quasitriangular structures and exactly two non-minimal quasitriangular structures with $k[G(A_{2p})]$ as the corresponding minimal quasitriangular sub-Hopf algebra. Moreover, none of the above-mentioned quasitriangular structures is triangular.*

Proof: Let $A = A_{qp}$. Suppose (A, R) is quasitriangular, and set $B = \mathrm{sp}_k\{R^{(1)}\}$ and $H = \mathrm{sp}_k\{R^{(2)}\}$. Then, B and H are sub-Hopf algebras of A of the same dimension and $B^{*cop} \cong H$. By Proposition 3.5, B and H cannot have dimension pq, since the unique sub-Hopf algebra of this dimension is commutative but not cocommutative. Thus, either $B = H = A$ or $B, H \subseteq k[G(A)]$. If $B = H = A$ then, by Lemma 3.7, $q = 2$, and A_{2p} admits exactly $2p-2$ minimal quasitriangular structures by Theorem 3.8. If $B, H \subseteq k[G(A)]$ then $q = 2$ by Lemma 3.9, and (A_{2p}, R_1) and (A_{2p}, R_3) are the two non-minimal quasitriangular structures A_{2p} admits. This completes the proof of the theorem. ∎

We end this paper by constructing another family of pq^2-dimensional biproducts. The method is the same as that of A_{qp}, but since we replace the cyclic group of order q^2: $L = \langle h \rangle$, by a direct product of two cyclic groups of order q: $M = \langle h \rangle \times \langle g \rangle$, the situation changes dramatically. This can already be seen in Section 2; whilst M always admits minimal triangular structures (Example 2.5), L never does (Theorem 2.3).

Let $p, q, m, \langle b \rangle, \eta$ be as before, but let $h^i g^j \cdot b^k = b^{km^i}$, that is, g acts trivially on b. Let $\theta(b) = h \cdot b$, and form $\{b_i^j\}$ as before. Then $k[\langle b \rangle]$ is a left $k[\langle h \rangle \times \langle g \rangle]$ comodule via $b_i^j \mapsto g^i \otimes b_i^j$. It is not hard to verify that $(k[\langle h \rangle \times \langle g \rangle], k[\langle b \rangle])$ is an admissible pair and hence that $\mathcal{A}_{qp} = k[\langle b \rangle] \times k[\langle h \rangle \times \langle g \rangle]$ is a Hopf algebra with multiplication, comultiplication and antipode as follows:

$$(28) \qquad (b_i^j \times h^n g^m)(b_{i'}^{j'} \times h^{n'} g^{m'}) = \eta^{ni'}(b_i^j b_{i'}^{j'} \times h^{n+n'} g^{m+m'}),$$

$$(29) \qquad \Delta(b_i^j \times h^n g^m) = \sum_{t=0}^{q-1} b_{i-t}^j \times h^n g^{t+m} \otimes b_t^j \times h^n g^m$$

and

$$(30) \qquad s(b_i^j \times h^n g^m) = \eta^{-ni}(s(b_i^j) \times h^{-n} g^{-m-i}).$$

Assume that k contains primitive pth and qth roots of unity, and let β be a generator of $k[\langle b\rangle]^*$ and α, γ be generators of $k[\langle h\rangle \times \langle g\rangle]$ such that $\langle \alpha, h\rangle = 1$, $\langle \alpha, g\rangle = \eta$, $\langle \gamma, g\rangle = 1$ and $\langle \gamma, h\rangle = \eta$. Then, $\mathcal{A}_{qp}$ is self-dual via the map $\beta^i \times \alpha^j\gamma^k \mapsto b^i \times h^j g^k$. The proof of that is similar to the proof of Proposition 3.1. When there is no ambiguity we identify $h^i g^j$ with $1 \times h^i g^j$.

PROPOSITION 3.12: *Let $\mathcal{A} = \mathcal{A}_{qp}$ and k be as before. Then:*

(1) $G(\mathcal{A}) = \langle h\rangle \times \langle g\rangle$ *(hence $\mathcal{A}_{qp} \not\cong \mathcal{A}_{qp}$ as Hopf algebras).*

(2) $\mathcal{A}$ *has $p-1$ simple sub-coalgebras of dimension q^2, namely* $\mathcal{A}_n^k = \mathrm{sp}_k\{b_i^k \times g^j | 0 \le i, j \le q-1\}$.

(3) $\mathcal{A}$ *is cosemisimple and semisimple.*

The proof of this proposition is similar to the proof of Proposition 3.3.

PROPOSITION 3.13: *Let $\mathcal{A} = \mathcal{A}_{qp}$ and k be as before. Then:*

(1) *The non-trivial sub-Hopf algebras of $\mathcal{A}$ are: $k[G]$ where G is a subgroup of $G(\mathcal{A})$, and $\mathcal{B} = \mathrm{sp}_k\{b^i \times g^j | 0 \le i, j \le q-1\}$. Moreover, $\mathcal{B}$ is the unique sub-Hopf algebra of $\mathcal{A}$ of dimension pq, and it is commutative.*

(2) $\mathcal{A} \cong \mathcal{A}^{cop}$ *for all p and q. Furthermore, if $f\colon \mathcal{A} \to \mathcal{A}^{cop}$ is an isomorphism of Hopf algebras, then f is determined by*

$$f(b_i^j \times g^l h^k) = \mu_{i,j}(s(b_i^{r(j)}) \times g^{-i+wk-l}h^k),$$

where $\mu_{i,j} \in k^*$, $1 \le r(j) \le (p-1)/q$ and $0 \le w \le q-1$.

(3) *Let $f\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of Hopf algebras. Then f is determined by*

$$f(g) = g, \quad f(h) = g^w h \quad \text{and} \quad f(b \times 1) = b^r \times 1$$

where $1 \le r \le p-1$ and $0 \le w \le q-1$. Moreover, $\mathrm{Aut}_{\mathrm{Hopf}}(\mathcal{A}) \cong Z_q \times Z_p^\times$.

Proof: (1) Let $\mathcal{B}$ be a non-trivial sub-Hopf algebra of $\mathcal{A}$. Using similar arguments to those used in the proof of Proposition 3.5 yields that $\dim \mathcal{B} = q, q^2$ or pq. Moreover, if $\dim \mathcal{B} = q$ then $\mathcal{B} = k[G]$ where G is a sub-group of $\langle g\rangle \times \langle h\rangle$ of order q, and if $\dim \mathcal{B} = q^2$ then $\mathcal{B} = k[\langle g\rangle \times \langle h\rangle]$. Suppose now that $\dim \mathcal{B} = pq$. Then $|G(\mathcal{B})| = q$. We first show that $\mathrm{Im}(\pi_{|\mathcal{B}}) = k[G(\mathcal{B})]$, where $\pi\colon \mathcal{A} \to k[G(\mathcal{A})]$ is the projection map. Clearly, $k[G(\mathcal{B})] \subseteq \mathrm{Im}(\pi_{|\mathcal{B}}) \subseteq k[G(\mathcal{A})]$. Furthermore, $\dim \mathrm{Im}(\pi_{|\mathcal{B}})$ divides $\dim \mathcal{B}$. Since $\dim \mathrm{Im}(\pi_{|\mathcal{B}}) = q$ or q^2, it follows that $\dim \mathrm{Im}(\pi_{|\mathcal{B}}) = q$, hence $\mathrm{Im}(\pi_{|\mathcal{B}}) = k[G(\mathcal{B})]$. Now, by Proposition 3.12, $\mathcal{B}$ is cosemisimple and hence a direct sum of simple sub-coalgebras of $\mathcal{A}$. We next show that $\mathcal{B}$ does not contain $\mathcal{A}_n^k$ for $n \neq 0$. Assume otherwise, and let $\mathcal{A}_n^k \subset \mathcal{B}$ for some k and $n \neq 0$. Then, $b_0^k \times h^n \in \mathcal{B}$, and hence by (20) and the above, $\pi(b_0^k \times h^n) = \varepsilon(b_0^k)h^n = h^n \in G(\mathcal{B})$. Therefore $\mathcal{A}_0^k = \mathcal{A}_n^k h^{-n} \subset \mathcal{B}$. In particular, $\sum_{i=0}^{q-1} b_i^k \times 1 = \sum_{i=0}^{q-1}(\lambda^{-i} \rightharpoonup t) \cdot c_k \times 1 = c_k \times 1 \in \mathcal{B}$, and hence $b^r \times 1 \in \mathcal{B}$ for all $0 \leq r \leq p-1$. Thus, $b_i^k \times 1 \in \mathcal{B}$ for *all* $1 \leq k \leq (p-1)/q$, and hence $b_i^k \times h^n \in \mathcal{B}$ for *all* $1 \leq k \leq (p-1)/q$, and we conclude that $\mathcal{A}_n^k \subset \mathcal{B}$ for all k and n. This implies that $\mathcal{B} = \mathcal{A}$, hence not of dimension pq. Finally, since $\mathcal{B}$ contains only simple sub-coalgebras among $\{\mathcal{A}_0^k\}$ and $G(\mathcal{B})$, and $\dim \mathcal{B} = pq$, it follows that $\mathcal{B} = k[\langle g \rangle] \oplus (\bigoplus_{k=0}^{(p-1)/q} \mathcal{A}_0^k) = \mathrm{sp}_k\{b^i \times g^j | 0 \leq i,j \leq q-1\}$. In particular, $\mathcal{B}$ is commutative and it is the unique sub-Hopf algebra of $\mathcal{A}$ of dimension pq. Note that $\mathcal{B}$ is isomorphic to the unique sub-Hopf algebra of A_{qp} of dimension pq, B.

(2) Let $f\colon \mathcal{A} \to \mathcal{A}^{cop}$ be an isomorphism of Hopf algebras. Then, using similar arguments to those used in the proof of Proposition 3.7 yields that f must be determined by

$$f(g) = g^{-1}, \quad f(h) = g^w h^l \quad \text{and} \quad f(b \times 1) = s(b^r \times \sum_{v=0}^{q-1} \alpha_{v,j} g^v)$$

where $\alpha_{v,j} = \langle \beta, b \rangle^{-j} \langle \beta^j, b_v^1 \rangle$ for $0 \leq j \leq p-1$, $1 \leq r \leq p-1$, $0 \leq w \leq q-1$ and $1 \leq l \leq q-1$. Since $f[(1 \times h)(b \times 1)] = f(1 \times h) f(b \times 1)$, we conclude, as in the proof of Proposition 3.7, that

$$b^{m^{-l+1}r} \langle \alpha^i, (\sum_{v=0}^{q-1} \alpha_{v,j} g^v)^m \rangle = b^r \langle \alpha^i, \sum_{v=0}^{q-1} \alpha_{v,j} g^v \rangle$$

for all $0 \leq i \leq q-1$. Thus, $l = 1$ and $j = 0$, and we have

$$f(g) = g^{-1}, \quad f(h) = g^w h \quad \text{and} \quad f(b \times 1) = s(b^r \times 1). \tag{31}$$

At this point it is not hard to verify that f, given in (31), is indeed an isomorphism of Hopf algebras for any $0 \leq w \leq q-1$ and $1 \leq r \leq p-1$. Finally, we compute

$$\begin{aligned} f(b_i^j \times 1) &= f[(\lambda^{-i} \rightharpoonup t) \cdot c_j \times 1] \\ &= f(\sum_{k=0}^{q-1} \eta^{-ik} c_j^{m^k} \times 1) = \sum_{k=0}^{q-1} \eta^{-ik} f(c_j^{m^k} \times 1) \\ &= s(\sum_{k=0}^{q-1} \eta^{-ik} c_j^{rm^k} \times 1) \\ &= s((\lambda^{-i} \rightharpoonup t) \cdot c_j^r \times 1) \\ &= s((\lambda^{-i} \rightharpoonup t) \cdot \theta^{l(j)}(c_{r(j)}) \times 1) \\ &= \langle \lambda^i, \theta^{l(j)} \rangle s((\lambda^{-i} \rightharpoonup t) \cdot c_{r(j)} \times 1) \\ &= \mu_{i,j} s(b_j^{r(j)} \times 1) = \mu_{i,j}(s(b_j^{r(j)}) \times g^{-i}) \end{aligned}$$

where $c_{r(j)}$ is the representative of the orbit containing c_j^r, and $\mu_{i,j} = \langle \lambda^i, \theta^{l(j)} \rangle = \eta^{il(j)}$.

(3) Let f be an automorphism of $\mathcal{A}$. Then, using similar arguments to those used before yields that f must be determined by

$$f(g) = g, \quad f(h) = g^w h \quad \text{and} \quad f(b \times 1) = b^r \times 1$$

where $1 \leq r \leq p-1$ and $0 \leq w \leq q-1$. It is not hard to verify that f described in the last equation determines an automorphism of $\mathcal{A}$. Denote this map by $f_{w,r}$. Since $f_{w,r} \circ f_{u,t} = f_{w+u,rt}$ the result follows. This concludes the proof of the proposition. ∎

As a corollary we have the following:

THEOREM 3.14: *Suppose that the field k contains primitive pth and qth roots of unity and let $\mathcal{A} = \mathcal{A}_{qp}$. Then, $\mathcal{A}$ is minimal quasitriangular if and only if $q = 2$. Furthermore, the map $f_{w,r}$: $\mathcal{A}_{2p}^{*cop} \to \mathcal{A}_{2p}$ given by $f_{w,r}(\beta_i^j \times \alpha^k \gamma^l) = s(b_i^{jr}) \times g^{-i+wk-l} h^k$ determines a minimal quasitriangular, but not triangular, structures on $\mathcal{A}_{2p}$ for any $0 \leq w \leq 1$ and $1 \leq r \leq p-1$.*

Proof: By (31) and the fact that $\mathcal{A}$ is self-dual, we need only to check whether the map f: $\mathcal{A}^{*cop} \to \mathcal{A}$ given by

$$f(\beta_i^j \times \alpha^k \gamma^l) = \mu_{i,j}(s(b_i^{r(j)}) \times g^{-i+wk-l} h^k)$$

satisfies (QT.5)$'$. Indeed, let $p = \beta_i^j \times \alpha^k\gamma^l$ and $a = b_v^u \times g^d h^t$. Then, on one hand

$$\begin{aligned}
&\sum \langle p_{(1)}, a_{(2)}\rangle a_{(1)} f(p_{(2)}) \\
&= \sum_{x,y=0}^{q-1} \langle \beta_{i-x}^j, b_y^u\rangle \eta^{kd+(l+x)t}(b_{v-y}^u \times h^t g^{d+y}) f(\beta_x^j \times \alpha^k\gamma^l) \\
&= \sum_{x+y=i} \langle \beta_{i-x}^j, b_y^u\rangle \eta^{kd+(l+x)t}(b_{v-y}^u \times h^t g^{d+y}) \mu_{x,j}(s(b_x^{r(j)}) \times g^{-x+wk-l}h^k) \\
&= \sum_{x+y=i} \langle \beta_{i-x}^j, b_y^u\rangle \eta^{kd+(l+x)t+tx} \mu_{x,j}(b_{v-y}^u s(b_x^{r(j)}) \times g^{d+y-x+wk-l}h^{k+t}) \\
&= \sum_{y=0}^{q-1} \langle \beta_y^j, b_y^u\rangle \eta^{kd+lt+2t(i-y)} \mu_{i-y,j}(b_{v-y}^u s(b_{i-y}^{r(j)}) \times g^{d+2y-i+wk-l}h^{k+t})
\end{aligned}$$

while on the other hand

$$\begin{aligned}
&\sum \langle p_{(2)}, a_{(1)}\rangle f(p_{(1)}) a_{(2)} \\
&= \sum_{x,y=0}^{q-1} \langle \beta_x^j, b_{v-y}^u\rangle \eta^{k(d+y)+lt} f(\beta_{i-x}^j \times \alpha^k\gamma^{l+x})(b_y^u \times h^t g^d) \\
&= \sum_{x+y=v} \langle \beta_x^j, b_{v-y}^u\rangle \eta^{k(d+y)+lt} \mu_{i-x,j}(s(b_{i-x}^{r(j)}) \times g^{-i+x+wk-l-x}h^k)(b_y^u \times h^t g^d) \\
&= \sum_{x+y=v} \langle \beta_x^j, b_{v-y}^u\rangle \eta^{k(d+y)+lt+ky} \mu_{i-x,j}(b_y^u s(b_{i-x}^{r(j)}) \times g^{d-i+wk-l}h^{k+t}) \\
&= \sum_{x=0}^{q-1} \langle \beta_x^j, b_x^u\rangle \eta^{kd+lt+2k(v-x)} \mu_{i-x,j}(b_{v-x}^u s(b_{i-x}^{r(j)}) \times g^{d-i+wk-l}h^{k+t}).
\end{aligned}$$

Therefore, f satisfies (QT.5)$'$ if and only if

$$\begin{aligned}
&\sum_{y=0}^{q-1} \langle \beta_y^j, b_y^u\rangle \eta^{2t(i-y)} \mu_{i-y,j}(b_{v-y}^u s(b_{i-y}^{r(j)}) \times g^{2y}) \\
&\qquad = \sum_{x=0}^{q-1} \langle \beta_x^j, b_x^u\rangle \eta^{2k(v-x)} \mu_{i-x,j}(b_{v-x}^u s(b_{i-x}^{r(j)}) \times 1)
\end{aligned}$$

for all j, u, t, i, v and k. Clearly, if $q = 2$ then the above equality holds. If $q \neq 2$, then for $i = v = y \neq 0$ we have that if f satisfies (QT.5)$'$

$$\langle \beta_i^j, b_i^u\rangle \mu_{0,j}(b_0^u s(b_0^{r(j)}) \times g^{2y}) = \langle \beta_i^j, b_i^u\rangle \mu_{0,j}(b_0^u s(b_0^{r(j)}) \times 1).$$

Since, by Remark 3.2, for any j there exists u so that $\langle \beta_i^j, b_i^u\rangle \neq 0$, $\mu_{0,j} \neq 0$ and $b_0^u s(b_0^{r(j)}) \neq 0$, we conclude that f does not satisfy (QT.5)$'$.

Finally, we show that $f = f_{w,r}$ does not determine a triangular structure on $\mathcal{A} = \mathcal{A}_{2p}$. We first compute

$$\begin{aligned}
\langle \beta_i^j \times \alpha^k\gamma^l, u\rangle &= \sum \langle \beta_i^j \times \alpha^k\gamma^l, s(R^{(2)})R^{(1)}\rangle \\
&= \sum \langle \beta_{i-t}^j \times \alpha^k\gamma^{l+t}, s(R^{(2)})\rangle \langle \beta_t^j \times \alpha^k\gamma^l, R^{(1)}\rangle \\
&= \sum \langle \beta_{i-t}^j \times \alpha^k\gamma^{l+t}, s(f(\beta_t^j \times \alpha^k\gamma^l))\rangle \\
&= \sum \langle \beta_{i-t}^j \times \alpha^k\gamma^{l+t}, s(s(b_t^{jr}) \times g^{-t+wk-l}h^k)\rangle \\
&= \sum \eta^{-kt}\langle \beta_{i-t}^j \times \alpha^k\gamma^{l+t}, b_t^{jr} \times g^{-wk+l}h^{-k}\rangle \\
&= \sum (-1)^{kt}\langle \beta_{i-t}^j, b_t^{jr}\rangle (-1)^{k(l-wk)-k(l+t)} \\
&= \delta_{i,0}\langle \beta_0^j, b_0^{jr}\rangle (-1)^{wk^2}.
\end{aligned}$$

Recall that $\beta_0^j = \frac{1}{2}(\beta^j + \beta^{-j})$ and $b_0^{jr} = \frac{1}{2}(b^{jr} + b^{-jr})$. Hence, $\langle \beta_0^j, b_0^{jr}\rangle = \frac{1}{2}(\omega^{rj^2} + \omega^{-rj^2})$ where ω is a primitive pth root of unity. Therefore,

$$\langle \beta_i^j \times \alpha^k\gamma^l, u\rangle = \delta_{i,0}\tfrac{1}{2}(\omega^{rj^2} + \omega^{-rj^2})(-1)^{wk}.$$

But if $\mathcal{A}$ were triangular u would be a central grouplike element (since $s^2 = \mathrm{id}$), and hence equals $1 \times g^s$ for some $0 \le s \le 1$, and

$$\langle \beta_i^j \times \alpha^k\gamma^l, 1 \times g^s\rangle = \delta_{i,0}(-1)^{ks}.$$

In particular we would have that $(-1)^{wk}\frac{1}{2}(\omega^{rj^2} + \omega^{-rj^2}) = (-1)^{ks}$ for all k and s, and hence that $\frac{1}{2}(\omega^{rj^2} + \omega^{-rj^2}) = \pm 1$. We conclude the proof by showing that $\frac{1}{2}(\omega^{rj^2} + \omega^{-rj^2}) \neq \pm 1$. Indeed, $\frac{1}{2}(\omega^{rj^2} + \omega^{-rj^2}) = \pm 1$ if and only if $\omega^{rj^2} = \pm 1$ for all j. Since $3 \le p$ is prime we are done. ∎

In the following theorem we prove that, unlike A_{qp}, $\mathcal{A}_{qp}$ is quasitriangular for any p and q. Moreover, it is triangular with $u = 1$.

THEOREM 3.15: *Suppose that the field k contains primitive pth and qth roots of unity and let $\mathcal{A} = \mathcal{A}_{qp}$. Then the map $f\colon \mathcal{A}^{*cop} \to \mathcal{A}$ defined by $f(\beta_i^j \times \alpha^k\gamma^t) = \varepsilon(\beta_i^j)(1 \times h^{-k}g^t)$ determines a triangular structure on $\mathcal{A}$ with $u = 1$.*

Proof: It is not hard to verify that f is a Hopf algebra map. We show that f satisfies (QT.5)$'$. Let $a = b_i^m \times h^k g^l$ and $p = \beta_j^n \times \alpha^r\gamma^s$. Then using (20), (28) and (29) we compute

$$\begin{aligned}
&\sum \langle p_{(1)}, a_{(2)}\rangle a_{(1)} f(p_{(2)}) \\
&\quad = \sum_{u,t=0}^{q-1} \langle \beta_u^n \times \alpha^r\gamma^{j-u+s}, b_{i-t}^m \times h^k g^l\rangle (b_t^m \times h^k g^{i-t+l})\varepsilon(\beta_{j-u}^n)(1\times h^{-r}g^s) \\
&\quad = \sum_{t=0}^{q-1} \langle \beta_j^n \times \alpha^r\gamma^s, b_{i-t}^m \times h^k g^l\rangle \varepsilon(\beta_0^n)(b_t^m \times h^{k-r} g^{i-t+l+s}) \\
&\quad = \langle \beta_j^n, b_j^m\rangle \eta^{rl+sk}\varepsilon(\beta_0^n)(b_{i-j}^m \times h^{k-r}g^{j+l+s})
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
&\sum \langle p_{(2)}, a_{(1)}\rangle f(p_{(1)}) a_{(2)} \\
&\quad = \sum_{u,t=0}^{q-1} \langle \beta_u^n \times \alpha^r\gamma^s, b_{i-t}^m \times h^k g^{l+t}\rangle \varepsilon(\beta_{j-u}^n)(1\times h^{-r}g^{s+u})(b_t^m \times h^k g^l) \\
&\quad = \sum_{t=0}^{q-1} \langle \beta_j^n \times \alpha^r\gamma^s, b_{i-t}^m \times h^k g^{l+t}\rangle \varepsilon(\beta_0^n)\eta^{-rt}(b_t^m \times h^{k-r} g^{l+s+j}) \\
&\quad = \langle \beta_j^n, b_j^m\rangle \eta^{-r(i-j)+sk+r(l+i-j)}\varepsilon(\beta_0^n)(b_{i-j}^m \times h^{k-r}g^{j+l+s}) \\
&\quad = \langle \beta_j^n, b_j^m\rangle \eta^{rl+sk}\varepsilon(\beta_0^n)(b_{i-j}^m \times h^{k-r}g^{j+l+s}).
\end{aligned}$$

Therefore f determines a quasitriangular structure on $\mathcal{A}$. We conclude the proof of the theorem by showing that $u = 1$ (hence, in particular, the structure is triangular). Indeed,

$$\begin{aligned}
\langle \beta_i^j \times \alpha^r\gamma^s, u\rangle &= \sum \langle (\beta_i^j \times \alpha^r\gamma^s)_{(1)}, s\circ f((\beta_i^j \times \alpha^r\gamma^s)_{(2)})\rangle \\
&= \sum_{t=0}^{q-1} \langle \beta_{i-t}^j \times \alpha^r\gamma^{s+t}, s\circ f(\beta_t^j \times \alpha^r\gamma^s)\rangle \\
&= \sum_{t=0}^{q-1} \langle \beta_{i-t}^j \times \alpha^r\gamma^{s+t}, s(\varepsilon(b_t^j)1 \times h^{-r}g^s)\rangle \\
&= \langle \beta_i^j \times \alpha^r\gamma^s, 1\times h^r g^{-s}\rangle \\
&= \langle \beta_i^j, 1\rangle \eta^{-rs+rs} = \langle \beta_i^j \times \alpha^r\gamma^s, 1\times 1\rangle.
\end{aligned}$$

This concludes the proof of the theorem. ∎

We summarize:

THEOREM 3.16: *Let p and q be prime numbers satisfying $p = 1 \pmod q$, and let k be a field containing primitive pth and qth roots of unity. Then $\mathcal{A}_{qp}$ is a self-dual semisimple Hopf algebra of dimension pq^2 which is not isomorphic to A_{qp}. Moreover, $\mathcal{A}_{qp}$ admits a non-minimal triangular structure, with $k[G(\mathcal{A}_{qp})]$ as the corresponding minimal triangular sub-Hopf algebra, for any p and q. Furthermore, $\mathcal{A}_{qp}$ admits minimal quasitriangular structures if and only if $q = 2$, and $\mathcal{A}_{2p}$ admits exactly $2p - 2$ such structures none of which is triangular.*

Remark 3.17: The referee has pointed out that the Hopf algebras A_{qp} and $\mathcal{A}_{qp}$ can be constructed in a unified way. Suppose the base field k contains enough roots of unity. Let U be a finite cyclic group which acts from the right on a Hopf algebra B (as Hopf algebra automorphisms). Identifying $(kU)^* = kU$ as usual, B is a left kU-comodule Hopf algebra via $x \mapsto \sum_u e_u \otimes x \cdot u$, where e_u is the dual basis of u $(\in U)$. Construct the smash coproduct $K = B\#kU$ with respect to this action. Since kU is commutative, K is a Hopf algebra with the algebra structure of tensor product. Let $V = \langle h \rangle$ be a cyclic group of order n, which acts from the left on B so that $(v \cdot x) \cdot u = v \cdot (x \cdot u)$ for $u \in U$, $v \in V$ and $x \in B$. Let V act on kU trivially. Then V acts on the Hopf algebra K. Construct the crossed product $A = K * V$ with respect to the action just defined and the relation $(1 * h)^n = (u_0 * 1)$, a fixed element in U. Then, one sees easily that $A (= K \otimes kV)$ is a Hopf algebra with the coalgebra structure of tensor product.

Suppose in addition that B is a group Hopf algebra of a finite cyclic group G, $U = V$ $(= \langle h \rangle)$ and the left action and the right action of U $(= V)$ on B coincide. Then it is easy to prove that A is self-dual (cf. Proposition 3.1).

In particular, let $G = \langle b \rangle$ be a cyclic group of order p, $n = q$ and $h \cdot b = b^m = b \cdot h$, where p, q and m are as above. If $u_0 = h$, then $A = A_{qp}$. If $u_0 = 1$, then $A = \mathcal{A}_{qp}$.

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